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Partial Sums of Lagged Cross-Products of AR Residuals and a Test for White Noise*

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Abstract

Partial sums of lagged cross-products of AR residuals are defined. By studying the sample paths of these statistics, changes in residual dependence can be detected that might be missed by statistics using only the total sum of cross-products. Also, a test statistic for white noise is proposed. It is shown that the limiting distribution of the test statistic converges weakly to a vector Brownian motion with independent elements under the null hypothesis of no residual autocorrelation. An indication of the circumstances under which the asymptotic results apply in finite-sample situations is obtained through a simulation study. Some considerations are given to the empirical size and power of the test statistic vis-à-vis the Ljung-Box (1978) portmanteau statistic, and a diagnostic test statistic proposed by Peña and Rodriguez (2002). An empirical example illustrates the importance of examining partial sums of time series residuals when inadequacies in model fit are anticipated due to a change in autocorrelation structure.

Key Words: Brownian motion, moncentral chi-square, partial sums, portmanteau diagnostic check, time series.

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1 Introduction

Let $\{X_t\}$ be a stationary, discrete time, scalar process. Suppose $\{X_t\}$ is generated by the AR model

$$X_t - \sum_{j=1}^{\infty} \phi_j X_{t-j} = \varepsilon_t, \quad t = 0, \pm 1, \ldots \tag{1}$$

where $\{\varepsilon_t\}$ are *i.i.d.* random variables such that $E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = \sigma^2 < \infty$ and $Cum(\varepsilon_t) = \kappa_4$, $|\kappa_4| < \infty$.[1] We also require that the function $A(z) = 1 - \sum_{j=1}^{\infty} \phi_j z^j$ be analytic on an open neighborhood of the closed unit disk $\mathcal{D}$ in the complex plane and has no zeroes on $\mathcal{D}$. Now, given $n + p^*$ observations $\{X_t; -p^* + 1 \leqslant t \leqslant n\}$, the ϕ_j's in (1) can be estimated by fitting a long AR(p^*) model to the data, where $p^* \equiv p_n$ is a sequence of positive integers depending on the sample size n. The least squares estimate $\hat{\boldsymbol{\phi}}_{p^*} = (\hat{\phi}_1, \ldots, \hat{\phi}_{p^*})'$ of $\boldsymbol{\phi}_{p^*} = (\phi_1, \ldots, \phi_{p^*})'$ is given by $\hat{\boldsymbol{\phi}}_{p^*} = (\sum_{i=p^*+1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1})^{-1} \sum_{i=p^*+1}^{n} \boldsymbol{X}_{i-1} X_i$ where $\boldsymbol{X}_i = (X_i, \ldots, X_{i-p^*+1})'$. The corresponding sequence of residuals $\{\hat{\varepsilon}_t\}$ is defined by $\hat{\varepsilon}_i = X_i - \hat{\boldsymbol{\phi}}'_{p^*}\boldsymbol{X}_{i-1}$, $(i = 1, \ldots, n)$. The initial values $\hat{\varepsilon}_{-p^*+1}, \ldots, \hat{\varepsilon}_0$ are set to zero if $p^* > 0$.

Given the above setup, one of the first questions that one can ask about the residual time series is simply "Is it white noise?". More formally, the problem of interest is to test the hypothesis of zero autocorrelation up to order m, i.e. $H_0 : \gamma_k/\gamma_0 = 0$, $(k = 1, 2, \ldots, m)$, versus the alternative hypothesis $H_1 : \gamma_k/\gamma_0 \neq 0$ for at least one k, where $\gamma_k = E(\varepsilon_t \varepsilon_{t-k})$ $(|k| < \infty)$. Common practice is to use a function of the lag k sample autocovariance, say $\hat{R}(n,k) = \sum_{t=k+1}^{n} \hat{\varepsilon}_t \hat{\varepsilon}_{t-k}$. Examples are the Box-Pierce (BP) test statistic, defined as $Q_m^{\text{BP}} = \sum_{k=1}^{m} n\{\hat{R}(n,k)/\hat{R}(n,0)\}^2$, and the Ljung-Box (LB), defined as $Q_m^{\text{LB}} = \sum_{k=1}^{m}\{n(n+2)/(n-k)\}\{\hat{R}(n,k)/\hat{R}(n,0)\}^2$.

Clearly, both the BP and the LB statistics only use the end points of a residual series, representing the total sums $\hat{R}(n,k)$. In this paper we advocate the use of partial sums of lagged cross-products of lag $k < n$ residuals, defined as

$$\hat{R}(j,k) = \sum_{t=k+1}^{j} \hat{\varepsilon}_t \hat{\varepsilon}_{t-k}, \quad (k+1 \leqslant j \leqslant n; n > 1). \tag{2}$$

By studying the sample paths of $\hat{R}(j,k)$, for various values of j, more subtle changes in residual correlation structure may be detected; see Section 5 for an example. Also, based on (2), various

[1] Formally, the cumulants $\kappa_1, \kappa_2, \ldots$ are defined by the identity in t $\exp\{\sum_{\nu=1}^{\infty} \kappa_\nu (it)^\nu/\nu!\} = \phi(t)$ with $\phi(t)$ the characteristic function. Subject to conditions of existence of moments, κ_ν can be expressed in terms of the central moments μ_ν. For instance, $\kappa_1 = 0$, $\kappa_2 = \mu_2$, $\kappa_3 = \mu_3$, and $\kappa_4 = \mu_4 - 3\mu_2^2$. The cumulant of order ν exists if μ_ν and lower μ's exist.

omnibus test statistics for white noise can be defined. In the following we focus on the test statistic

$$n^{-2}\hat{Q}_m = \sum_{k=1}^{m}\sum_{j=k+1}^{n}\Big\{\hat{R}(j,k)/\hat{R}(n,0)\Big\}^2. \tag{3}$$

Notice that, when $j = n$, the expression on the right hand side of (3) becomes $n^{-1}Q_m^{\mathrm{BP}}$. Hence, the test statistic $n^{-2}\hat{Q}_m$ incorporates information contained in the jth $(k+1 \le j \le n)$ subset of the residual series.

MacNeill (1978) studied the limiting process of partial sums of, fixed-design, regression residuals. MacNeill and Jandhyala (1985) extended these results to non-linear regressions. Bai (1993) considered the limiting process of partial sums of residuals in stationary and invertible ARMA models. Yu (2007) considered high moment partial sum processes for residuals obtained from stationary ARMA models, i.e. $\sum_{t=k+1}^{j}\hat{\varepsilon}_t^i$ $(i = 2,3,4)$. Also, Kulperger and Yu (2005) constructed high moment partial sums processes for residuals from GARCH models. Most of these works have benefited from the literature (see, e.g., Csörgő and Horváth (1997) for an extensive review) on CUSUM test statistics, often used for testing model structure changes via cumulated sums of squared regression residuals $\hat{R}(j,0)$.

The properties of $\hat{R}(j,k)$ were studied by De Gooijer and MacNeill (1999) for fixed-design regression models. They showed that the resulting Gaussian process is characterized in terms of (i) regressor functions, (ii) the serial correlation structure of the noise process, (iii) the distribution of the noise process which, for linear time series, reduces to the 4th-order cumulant of the error process, and (iv) the order of the lags of the cross-product residuals. These four factors affect the limiting process of $\hat{R}(j,k)$ independently.

The plan of the paper is as follows. In Section 2, we first derive the asymptotic null distribution of $\sum_{j=k+1}^{n}\{\hat{R}(j,k)/\hat{R}(n,0)\}^2$. Next, in Section 3, we consider the finite-sample distribution of $\sum_{k=1}^{m}\sum_{j=k+1}^{n}\{R(j,k)/R(n,0)\}^2$, where $R(j,k) = \sum_{t=k+1}^{j}\varepsilon_t\varepsilon_{t-k}$, a quantity closely related to (2). Using moment properties of this quantity, the large-sample distribution of $n^{-2}\hat{Q}_m$ is established in Section 4. We also compare, via a small-scale simulation study, the empirical size and power of $n^{-2}\hat{Q}_m$ vis-à-vis Q_m^{LB} and a portmanteau test statistic recently proposed by Peña and Rodriguez (2002); see Section 4 for details. In Section 5, we demonstrate that the $\hat{R}(j,k)$'s are a useful tool in detecting a change in the residual correlation structure obtained from a long AR model fitted to a time series consisting of IBM closing stock prices. In addition, we enhance the usefulness of the proposed test statistic. Finally, Section 6 concludes.

2 Asymptotic null distribution of $\sum_{j=k+1}^{n}\{\hat{R}(j,k)/\hat{R}(n,0)\}^2$

To examine the test $n^{-2}\hat{Q}_m$, it is useful to consider initially the limit distribution of $R(j,k)$. In addition to the basic assumptions in Section 1, the asymptotic results require that $\sum_{\nu=-\infty}^{\infty}|\gamma_\nu| < \infty$. Then the spectral density function $f(\lambda) = (1/2\pi)\sum_{|\nu|<\infty} e^{-i\lambda\nu}\gamma_\nu$, $(\lambda \in [-\pi,\pi])$ exists. If the spectral density is positive, that is, if

$$f(\lambda) \geqslant a > 0, \quad \lambda \in [-\pi,\pi], \tag{4}$$

the process can be expressed either as (1) or as an MA(∞) representation, i.e. $X_t = \sum_{j=1}^{\infty} a_j\varepsilon_{t-j}$, where a_j are uniquely determined by the relation $A^{-1}(z) = \sum_{j=0}^{\infty} a_j z^j$ for $|z| \leq 1$. Then there exists constants $\rho \in (0,1)$ and $C > 0$, such that $\max\{|\phi_j|,|a_j|\} \leq C\rho^j$ for all $j \geq 0$.

We require a central limit theorem for time series. Conditions that guarantee convergence in distribution of $n^{-1/2}\sum_{j=1}^{[nt]}\varepsilon_j$, $t \in [0,1]$, to the normal with zero mean and variance $\{2\pi f(0)t\}$, are those given by Brillinger (1973). These conditions are stated in terms of cumulant functions, defined as follows: $C_{\kappa+1}(\nu_1,\nu_2,\ldots,\nu_\kappa) = \mathrm{Cum}\{\varepsilon_{j+\nu_1},\varepsilon_{j+\nu_2},\ldots,\varepsilon_{j+\nu_\kappa},\varepsilon_j\}$. Stationarity to order $\kappa+1$ is implicit in this definition. When necessary we assume the cumulants exist. Moreover, we assume that for some finite L_κ, $\kappa = 1,2,\ldots$, the mixing condition (Brillinger, 1973, Assumption I)

$$|C_{\kappa+1}(\nu_1,\nu_2,\ldots,\nu_\kappa)| < L_\kappa / \prod_{j=1}^{\kappa}(1+\nu_j^2) \tag{5}$$

is fulfilled. Assumption (5) can be weakened, for instance, by an α-mixing assumption.

Consider the zero-mean process $\{\tilde{\chi}(t,k), t \in [0,1]\}$ defined as follows:

$$\tilde{\chi}(t,k) = \chi(t,k) - n^{1/2}(t - \frac{k}{n})\gamma_k, \quad (n \geq 1), \tag{6}$$

where $n^{1/2}\chi(t,k) = R([nt],k) + (nt-[nt])^2\varepsilon_{[nt]+1}\varepsilon_{[nt]+1-k}$, with $[nt]$ denoting the integer part of nt. Let $K(s,t|k,n) = Cov\{\tilde{\chi}(s,k),\tilde{\chi}(t,k)\}$ denote the covariance kernel of the process (6). De Gooijer and MacNeill (1999) showed that, uniformly in s and t, $K(s,t|k,n) \to K(k)\min(s,t)$, where

$$K(k) = 2\pi\int_{-\pi}^{\pi}(1+e^{2i\lambda k})f^2(\lambda)d\lambda + F(k) \tag{7}$$

and where $F(k) = \sum_{|\nu_i|<\infty}\mathrm{Cum}(\varepsilon_i,\varepsilon_{i+k},\varepsilon_{i+\nu},\varepsilon_{i+k+\nu})$. Moreover, under assumptions (4) and (5) and provided the spectral density of the noise process is square integrable, the m-vector $(\tilde{\chi}(t_1,k),\tilde{\chi}(t_2,k),\ldots,\tilde{\chi}(t_m,k))$ has a non-trivial asymptotic probability distribution that is normal with zero mean and covariance matrix $\parallel K(k)\min(t_i,t_j) \parallel$. Next, define the process

$\{B_\epsilon,\ t \in [0,1]\}$ as $B_\epsilon(t) = \{K(k)\}^{1/2} B(t)$ with $\{B(t), t \in [0,1]\}$ the standard Brownian motion process. Assume W_ϵ is the measure in $C[0,1]$ corresponding to $B_\epsilon(\cdot)$. Then, under assumptions (4), (5), and provided the covariance function of the noise process is square integrable, it is easy to show that $P_{\epsilon_n} \Longrightarrow W_\epsilon$, where '$\Rightarrow$' denotes weak convergence in distribution. Thus the process (6) converges weakly to Brownian motion.

Now consider the limit process of $\hat{R}(j,k)$. First, we define the following two sequences of stochastic processes, both possessing continuous sample paths,

$$
\begin{aligned}
n^{1/2}\theta(t,k) &= \hat{R}([nt],k) + (nt-[nt])^2 \hat{\varepsilon}_{[nt]+1}\hat{\varepsilon}_{[nt]+1-k}, \quad t \in [0,1], && (8)\\
D(t,k) &= n^{1/2}\{\chi(t,k) - \theta(t,k)\}, \quad t \in [0,1]. && (9)
\end{aligned}
$$

Let $\boldsymbol{I}_n$ denote an $n \times n$ matrix and $\boldsymbol{I}_{n-k}(t)$ an $n \times n$ matrix with the first $[nt]-k$ diagonal elements equal to 1, the next equal to $nt-[nt]$, and all other elements equal to zero. Also, let $\boldsymbol{I}_{k,n}(t)$ be the $n \times n$ matrix given by

$$
\boldsymbol{I}_{k,n}(t) = \begin{pmatrix} & \boldsymbol{O}_{k\times n} & \\ \boldsymbol{I}_{[nt]} & & \boldsymbol{O}_{[nt]\times(n-[nt])} \\ \boldsymbol{O}_{(n-k-[nt])\times[nt]} & & \boldsymbol{U}_{(n-k-[nt])\times(n-[nt])} \end{pmatrix},
$$

with $\boldsymbol{O}_{a\times b}$ denoting an $a \times b$ null matrix, and $\boldsymbol{U}_{(n-k-[nt])\times(n-[nt])}$ is an $(n-k-[nt]) \times (n-[nt])$ matrix with the element in the first entry equal to $nt-[nt]$ and the remaining elements equal to zero. Then (9) can be rewritten as

$$
\begin{aligned}
D(t,k) &= (\hat{\boldsymbol{\phi}}_{p^*} - \boldsymbol{\phi}_{p^*})' \boldsymbol{X}_n' \boldsymbol{I}_{k,n}(t) \boldsymbol{I}_{n-k}(t) \boldsymbol{\varepsilon}_n + \boldsymbol{\varepsilon}_n' \boldsymbol{I}_{k,n}(t) \boldsymbol{I}_{n-k}(t) \boldsymbol{X}_n (\hat{\boldsymbol{\phi}}_{p^*} - \boldsymbol{\phi}_{p^*}) \\
&\quad -(\hat{\boldsymbol{\phi}}_{p^*} - \boldsymbol{\phi}_{p^*})' \boldsymbol{X}_n' \boldsymbol{I}_{k,n}(t) \boldsymbol{I}_{n-k}(t) \boldsymbol{X}_n (\hat{\boldsymbol{\phi}}_{p^*} - \boldsymbol{\phi}_{p^*}) \\
&= H_{t_1}\{\theta(t,k)\} + H_{t_2}\{\theta(t,k)\} + H_{t_3}\{\theta(t,k)\}, \quad \text{say},
\end{aligned}
$$

where $H_{t_i}\{\cdot\}$ $(i=1,2,3)$ are continuous functions of $\theta(t,k)$.

To assert that the large-sample distribution theory for $\{\tilde{\theta}(t,k), t \in [0,1]\}$ is the same as that for $\{\tilde{\chi}(t,k), t \in [0,1]\}$, where $\tilde{\theta}(t,k) = \theta(t,k) - E\{\theta(t,k)\}$, we need the following result, contained in Lee and Wei (1999). Suppose that p^* satisfies

$$
n^{-1/2}(p^*)^2 \log n \to 0 \quad \text{and} \quad n^{7/4} p^* \rho^{p^*} \to 0 \quad \text{for all} \quad \rho \in (0,1) \quad \text{as} \quad n \to \infty. \tag{10}
$$

Then, $\| \hat{\boldsymbol{\phi}}_{p^*} - \boldsymbol{\phi}_{p^*} \|^2 = O_p(n^{-1}p^*)$. Clearly, the first part of condition (10) requires p^* not to be so large that the AR model is overfitted whereas the second part implies that p^* should not be too small to avoid a meaningless approximation. A typical p^* satisfying (10) is $c(\log n)^2$ with $c > 0$, or un^v with $u > 0$, $0 < v < 1/4$.

In view of the above result, we have

$$\begin{aligned}\sup_{0\leqslant t\leqslant 1} \parallel \frac{1}{\sqrt{n}} H_{t_1}\{\theta(t,k)\} \parallel &\leqslant \parallel \hat{\boldsymbol{\phi}}_{p^*} - \boldsymbol{\phi}_{p^*} \parallel \frac{1}{\sqrt{n}} \sup_{0\leqslant t\leqslant 1} \parallel \boldsymbol{X'}_n \boldsymbol{I}_{k,n}(t) \boldsymbol{I}_{n-k}(t) \boldsymbol{X}_n \boldsymbol{\varepsilon}_n \parallel \\ &= O_p((p^*)^{3/2}/\sqrt{n}) = o_p(1).\end{aligned}$$

In a similar vein $\sup_{0\leqslant t\leqslant 1} \parallel \frac{1}{\sqrt{n}} H_{t_2}\{\theta(t,k)\} \parallel \leqslant o_p(1)$. Also, it is straightforward to see that

$$\sup_{0\leqslant t\leqslant 1} \parallel \frac{1}{\sqrt{n}} H_{t_3}\{\theta_{p^*}(t,k)\} \parallel \leqslant \parallel \hat{\boldsymbol{\phi}}_{p^*} - \boldsymbol{\phi}_{p^*} \parallel^2 \frac{1}{\sqrt{n}} \parallel \boldsymbol{X'}_n \boldsymbol{I}_{k,n}(t) \boldsymbol{\varepsilon}_n \boldsymbol{I}_{n-k}(t) \boldsymbol{X}_n \parallel = O_p((p^*)^2/\sqrt{n}) = o_p(1).$$

Thus, on combining these results, $n^{-1/2} \sup_{0\leqslant t\leqslant 1} |D(t,k)| \to 0$ with probability 1. Then, with a proper normalization, the invariance principle for partial sums of *i.i.d.* sequences $\{\varepsilon_t\}$ implies that

$$\frac{(\theta(t,k)/\theta(1,0)) - (t\gamma_k/\gamma_0)}{\{K(k)/\gamma_0^2\}^{1/2}} \Rightarrow B_k(t), \quad t \in [0,1],$$

where $\{B_k(t), t \in [0,1]\}$ is a zero-mean stationary Gaussian process. The covariance kernel $K(s,t|k) = Cov\{B_k(s), B_k(t)\}$ of the limit process is given by De Gooijer and MacNeill (1999). Hence, under Assumptions (4), (5) and those of (10), and provided the spectral density of the noise process is square integrable, we see that under H_0

$$\sum_{j=k+1}^{n} \{\hat{R}(j,k)/\hat{R}(n,0)\}^2 \Rightarrow \frac{K(k)}{\gamma_0^2} \int_0^1 B^2(t)dt. \tag{11}$$

Thus, the asymptotic null distribution of $\sum_{j=k+1}^{n} \{\hat{R}(j,k)/\hat{R}(n,0)\}^2$ depends on the nature of the data generating process $\{X_t\}$ through the factor $K(k)/\gamma_0^2$.

3 The statistic $n^{-2}T_m$

A statistic closely related to $n^{-2}\hat{Q}_m$ is given by

$$n^{-2}T_m = \sum_{k=1}^{m} \sum_{j=k+1}^{n} \{R(j,k)/R(n,0)\}^2. \tag{12}$$

The mean and variance of $n^{-2}T_m$ can be determined as follows. Defining $\boldsymbol{\varepsilon}_j = (\varepsilon_1, \ldots, \varepsilon_j)'$, we can write $\sum_{j=k+1}^{n} R(j,k) = \sum_{j=k+1}^{n} \boldsymbol{\varepsilon}'_j \boldsymbol{A}(j,k) \boldsymbol{\varepsilon}_j$ where $\boldsymbol{A}(j,k)$ is an $j \times j$ null matrix except for values 1/2 everywhere on the kth upper and lower diagonal. Since the limiting process of $\sum_{j=k+1}^{n} R(j,k)$ is normally distributed it is easy to see that $E\{R^2(n,0)\}^{r+s} = n(n+2)\cdots(n+2r+2s-2)$. After some algebra it follows that for finite values of n

$$E\{\sum_{j=k+1}^{n} R^2(j,k)\} = \frac{1}{2}(n-k)(n-k+1). \tag{13}$$

The mean of $n^{-2}T_m$ is now straightforward to evaluate, and is given by

$$E\{n^{-2}T_m\} = \frac{m}{2}\Big(1 + \frac{m^2-1}{6n} - \frac{m^2+6m+11}{6(n+2)}\Big). \tag{14}$$

Note that if n is large relative to m, (14) is equal to the sum of expected values of m independent one-dimensional standard Brownian motions, i.e. $m/2$. After some more algebra it can also be verified that

$$\begin{aligned} E\{(\sum_{j=k+1}^{n} R^2(j,k))^2\} &= \frac{1}{12}(40n - 64k - 358kn + 287k^2 - 326k^3 + 7k^4 + 119n^2 + 86n^3 + 7n^4 \\ &\quad - 378kn^2 + 594k^2n - 28kn^3 - 28k^3n + 42k^2n^2), \qquad (k < n/2), \end{aligned} \tag{15}$$

and

$$\begin{aligned} E\{\sum_{j=k+1}^{n} R^2(j,k) \sum_{j=\ell+1}^{n} R^2(j,\ell)\} &= \frac{1}{4}(n-k)(n-k+1)(n-\ell)(n-\ell+1) \\ &\quad + \frac{2}{3}(n-\ell)(6kn - 6k\ell - 8\ell n + 4\ell^2 + 4n^2 - 1) \\ + \frac{2}{3}(\ell - n + 1)(12k + 5\ell - 5n - 6k\ell + 6kn + 4\ell n - 6k^2 &- 2\ell^2 - 2n^2 - 6), \ (k \neq \ell, k < n/2). \end{aligned} \tag{16}$$

Expressions (15) and (16) were checked in several ways. For instance, by evaluating (15) for certain values of k and n, we obtained the same results as Sanjel, Provost and MacNeill (2005) who used a symbolic computational methodology.

The covariance between $\sum_{j=k+1}^{n}\{R^2(j,k)/R^2(n,0)\}$ and $\sum_{j=\ell+1}^{n}\{R^2(j,\ell)/R^2(n,0)\}$ is given by

$$\begin{aligned} Cov\Big\{\sum_{j=k+1}^{n}\frac{R^2(j,k)}{R^2(n,0)}, \sum_{j=\ell+1}^{n}\frac{R^2(j,\ell)}{R^2(n,0)}\Big\} &= \frac{E\{\sum_{j=k+1}^{n} R^2(j,k)\sum_{j=\ell+1}^{n} R^2(j,\ell)\}}{n(n+2)(n+4)(n+6)} \\ &\quad - \frac{(n-k)(n-k+1)(n-\ell)(n-\ell+1)}{4n^2(n+2)^2} \end{aligned} \tag{17}$$

with $E\{\sum_{j=k+1}^{n} R^2(j,k)\sum_{j=\ell+1}^{n} R^2(j,\ell)\}$ given in (16). One can show that the covariance is positive and that, for fixed values of m and ℓ, it goes to 0 as $n \to \infty$. The exact variance of $n^{-2}T_m$ follows from

$$Var\{n^{-2}T_m\} = \sum_{k=1}^{m} Var\Big\{\sum_{j=k+1}^{n}\frac{R^2(j,k)}{R^2(n,0)}\Big\} + 2\sum_{k=1}^{m-1}\sum_{\ell=k+1}^{m} Cov\Big\{\sum_{j=k+1}^{n}\frac{R^2(j,k)}{R^2(n,0)}, \sum_{j=\ell+1}^{n}\frac{R^2(j,\ell)}{R^2(n,0)}\Big\}, \tag{18}$$

where for fixed n, $Cov\{\sum_{j=k+1}^{n}\{R^2(j,k)/R^2(n,0)\}, \sum_{j=\ell+1}^{n}\{R^2(j,\ell)/R^2(n,0)\}\} \neq 0$. Using

(14), (15) and (17), the variance of $n^{-2}T_m$ for $k < n/2$ is given by

$$
\begin{aligned}
Var\{n^{-2}T_m\} = \frac{m}{3}\Big(1 &- \frac{m\,(m-1)^2\,(m+1)^2}{48n^2} + \frac{65m^5 - 12m^4 - 1480m^3 - 240m^2 - 25m + 252}{2880n} \\
&- \frac{m(m^2+6m+11)^2}{48(n+2)^2} - \frac{25m^5 - 192m^4 - 3380m^3 - 11580m^2 - 16625m + 252}{960(n+2)} \\
&+ \frac{5m^5 - 252m^4 - 4720m^3 - 26400m^2 - 63685m - 2148}{960(n+4)} \\
&- \frac{5m^5 - 192m^4 - 5500m^3 - 44700m^2 - 152725m + 10332}{2880(n+6)}\Big). \qquad (19)
\end{aligned}
$$

For n large relative to m, $Var\{n^{-2}T_m\}$ is equal to sum of the variances of m independent one-dimensional standard Brownian motions, i.e. $m/3$. A simple check on (14) and (19) is that the terms in the numerator should have coefficients that sum to zero which, after some algebra, can be verified. From (13), we see that T_m can be modified into the following unbiased statistic $T_m^* = \sum_{k=1}^m \{n(n+2)\}/\{(n-k)(n-k+1)\} \sum_{j=k+1}^n \{R(j,k)/R(n,0)\}^2$. The properties of this statistic will not be explored here.

Since $n^{-2}T_m$ is a sum of squares of ratios of quadratic forms in normal random variables it is reasonable to suppose that its distribution could be approximated by a noncentral χ^2 distribution. This distribution can be approximated using a Laguerre polynomial truncated at some high order. The simplest approximation consists of replacing the noncentral χ^2 distribution by a multiple of central χ^2, $a\chi_b^2$, say, where a and b are constants chosen so that the approximate distribution has the same mean and variance as $n^{-2}T_m$. This implies that a and b are determined by $a = Var\{n^{-2}T_m\}/2E\{n^{-2}T_m\}$ and $b = 2E^2\{n^{-2}T_m\}/Var\{n^{-2}T_m\}$; see, e.g., Patnaik (1949). Using Monte Carlo simulations (not shown here), we noted that this approximation to the exact distribution of $n^{-2}T_m$ is adequate for $n = 100$, 300, and various values of m.

4 The test statistic $n^{-2}\hat{Q}_m$

4.1 Large-sample distribution

The results of Sections 2 and 3 suggest that, if the model is correctly identified and fitted, the statistic $n^{-2}\hat{Q}_m$ will converge weakly to a vector Brownian motion with independent elements. That is, we have

$$\sum_{k=1}^{m}\Big(\frac{\sum_{j=k+1}^{n}(\hat{R}(j,k)/\hat{R}(n,0))}{\{K(k)/\gamma_0^2\}^{1/2}}\Big)^2 \Rightarrow \mathrm{tr}\{\int_0^1 \boldsymbol{B}(t)\boldsymbol{B}'(t)dt\}, \qquad (20)$$

where $\boldsymbol{B}(t)$ is the $m \times 1$ vector $(B(t), \ldots, B(t))'$. The characteristic function $\Psi_m(t)$ of $\mathrm{tr}\{\int_0^1 \boldsymbol{B}(t)$ $\boldsymbol{B}'(t)dt\}$ is given by $\prod_{n=1}^{\infty}\{1 - (2/(2n-1)\pi)^2(\sqrt{2it})^2\}^{-m/2}$. Using the Laplace transform of

$\Psi_m(t)$, one can derive the following expression for the limiting distribution function

$$\Omega_m(c) = 2^{(m+2)/2} \sum_{j=0}^{\infty} \binom{-m/2}{j} \Big(1 - \Phi\Big(\frac{4j+m}{2\sqrt{c}}\Big)\Big), \quad c > 0, \tag{21}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function. The proof of (21) is similar to the one given by Rothman and Woodroofe (1972) when $m = 1$. Selected quantiles of (21) are presented in Table 1. Given the relationship $\chi_1^2(x) = 2\Phi(\sqrt{x}) - 1$, it is clear from (21) that the limiting distribution of $n^{-2}\hat{Q}_m$ is a weighted sum of independent χ_1^2 random variables. The weights are the eigenvalues of a positive definite symmetric matrix.

If $\{\varepsilon_t\}$ follows a white noise process with $\kappa_4 = 0$, it is straightforward to see that $K(k) = 1$ $\forall k \geqslant 1$. However, if $\{\varepsilon_t\}$ is a stationary ARMA process expression (20) indicates that at each lag k, the large-sample distribution of $n^{-2}\hat{Q}_m$ must be adjusted by the factor $K(k)/\gamma_0^2 = 2\pi \int_{-\infty}^{\infty}(1+e^{2i\lambda k})f^2(\lambda)d\lambda/\gamma_0^2$, where the term $F(k)$ has been omitted from (7) because it converges to zero. For instance, with $\{\varepsilon_t\}$ AR(1) errors – that is $\varepsilon_t = \beta\varepsilon_{t-1} + u_t$ ($|\beta| < 1$), where u_t are *i.i.d.* random variables such that $E(u_t) = 0$, $E(u_t^2) = 1$, and $Cum(u_t) = 0$ – then $K(k)/\gamma_0^2 = \{(2k-1)\beta^{2k+2} - (2k+1)\beta^{2k} - \beta^2 - 1\}/(\beta^2 - 1)$, ($k \geqslant 0$). For $k = 1$ this factor ranges from 1.05 for $|\beta| = 0.1$ to 18.86 for $|\beta| = 0.9$. Thus, one should be careful in using the quantiles reported in Table 1 if the departure from the white noise model is too large.

Table 1 about here

4.2 Finite-sample distribution: empirical size

In Section 3, we noted that the finite-sample distributions of $n^{-2}T_m$ can be closely approximated by a $a\chi_b^2$ distribution. Thus, if the model has been correctly identified and fitted, we expect that the statistic $n^{-2}\hat{Q}_m$ can also be adequately approximated by a $a\chi_b^2$ distribution, with the number of degrees of freedom reduced to $b - p^*$ due to estimating the p^* AR parameters in the model. To study this in more detail, we conduct a small-scale simulation experiment, examining the empirical size and power (Section 4.3) of the test statistic.

For comparison purposes, we also present results for the statistic Q_m^{LB} and a new portmanteau diagnostic test statistic proposed by Peña and Rodriguez (2002) (hereafter referred to as PR). The latter statistic is defined by

$$D_m = n[1 - |\tilde{\boldsymbol{R}}_m|^{1/m}], \tag{22}$$

where $\tilde{\boldsymbol{R}}_m = (\tilde{R}(n, i-j)/\tilde{R}(n,0))$ $((i,j) = 1, \ldots, m+1)$ is an $(m+1)\times(m+1)$ symmetric residual autocorrelation matrix, with typical element $\tilde{R}(n, i-j)/\tilde{R}(n,0)$, constructed by utilizing the standardized residual autocorrelations $\tilde{R}(n,k)/\tilde{R}(n,0) = \{(n+2)/(n-k)\}^{1/2}\{\hat{R}(n,k)/\hat{R}(n,0)\}$. Under the null hypothesis that the ARMA(p,q) model is correctly identified and fitted, the distribution of D_m can be approximated by a gamma distribution, $\mathcal{G}(\alpha, \beta)$, where

$$\alpha = \frac{3m[(m+1) - 2(p+q)]^2}{2[2(m+1)(2m+1) - 12m(p+q)]} \quad \text{and} \quad \beta = \frac{3m[(m+1) - 2(p+q)]}{2(m+1)(2m+1) - 12m(p+q)}.$$

To examine the empirical significance levels of the three statistics, we generated 10,000 series of lengths $n = 100$ and 200 from various AR(1) processes with $\varepsilon_t \sim i.i.d.\, N(0,1)$. The significance levels of Q_m^{LB} are obtained by using percentiles of the χ^2_{m-p-q} distribution, and those of D_m by using the $\mathcal{G}(\alpha, \beta)$ approximation. As for the $n^{-2}\hat{Q}_m$ test, critical values are obtained from the $a\chi^2_{b-p^*}$ approximation with $p^* = \lfloor 2n^{1/5} \rfloor$. All models are estimated by the maximum likelihood method. To save space, only simulation results are reported for the 5% significance level. Qualitatively similar results were obtained for a 1% level; they are available upon request.

Figure 1 about here

Figure 1.a) shows the empirical sizes of the three test statistics in the case of a white noise process (no estimation) for $n = 100$, and $m = 2, 3, \ldots, 40$. Note that for all values of m, there is a close agreement between the empirical and nominal size of the test statistic $n^{-2}\hat{Q}_m$ (squares). However, for values of m larger than about 10 the empirical sizes of the Q_m^{LB} (D_m) statistic, are considerably higher (lower) than the nominal level. Figure 1.b) shows the estimated sizes of Q_m^{LB} (triangles) and D_m (dots) for $n = 100$. Here the data are generated from an AR(1) model with $\phi_1 = 0.3$, and the fitted process is an AR(1) model. Comparing Figures 1.a) and 1.b), we see that the accuracy of the size of the portmanteau tests Q_m^{LB} and D_m depends not only on the choice of m but also on the underlying parameter value.

Figures 1.c) and 1.d) show the estimated sizes of $n^{-2}\hat{Q}_m$ (squares), Q_m^{LB} (triangles) for $m = 6, 7, \ldots, 40$, and $n = 100$. The data are generated from an AR(1) model with parameters $\phi_1 = 0.3$ and 0.5, and the fitted model is an AR(5), so $p^* = 5$. The estimated sizes of D_m (dots) are computed with $m = 14, 15, \ldots, 40$.[2] We see that, for all selected parameter values ϕ_1, the D_m statistic has considerable size distortions when m is less than about 35. A similar result was also noticed for $n = 200$. These results contradict Peña and Rodriguez's (2002, p. 604) claim

[2]If m is too small it can happen that the gamma distribution has parameters $\alpha \leq 0$ or $\beta \leq 0$, which is numerically infeasible.

that "... The significance level of D_m does not seem to be affected by the value of m ...". For $m > 35$ the size properties of the $n^{-2}\hat{Q}_m$ and D_m statistics are about the same, but still not very satisfactory as compared to the Q_m^{LB} statistic. The latter test is a clear-cut winner in terms of achieving proper sizes for almost all values of ϕ_1 and m.

The "size distortion" problem of the D_m test has been noted by other researchers. In particular, Kwan and Wu (2003) performed an extensive simulation study of the finite-sample performance of the LB, PR, and Monti's (1994) portmanteau statistics in testing the adequacy of various AR(1), MA(1), and nonlinear time series models. Their results reveal that the estimated sizes of D_m obtained from the gamma approximation are substantially larger than the nominal level when m is small and the parameter values are close to the boundary of the stationary or invertible region. Also Lin and McLeod (2006) detected serious size distortions of the D_m test for AR(2) and ARMA$(1,1)$ models with $m = 10$.

4.3 Finite-sample distribution: empirical power

The empirical powers of $n^{-2}\hat{Q}_m$, Q_m^{LB}, and D_m were obtained for an AR(1) processes (no estimation). To conserve space, we summarize the main results. All three tests achieve similar powers when m is small ($m \leq 3$). This is reasonable because evidence of misspecifications is likely to occur in the first few residual autocorrelations. However, surprisingly, the estimated powers of D_m are much higher when $m > 3$. For the LB and PR test statistics these results are in agreement with power results reported by Kwan and Wu (2003).

Here we compare the empirical powers of $n^{-2}\hat{Q}_m$, Q_m^{LB}, and D_m for the following data generating process: $X_t = \phi_1 X_{t-1} + \varepsilon_t$ if $t = 1, \ldots, n/2$ and $X_t = -\phi_1 X_{t-1} + \varepsilon_t$ if $t = n/2+1, \ldots, n$ with $\varepsilon_t \sim i.i.d.\, N(0,1)$. The overall behaviour of the series generated by this process reflects the behaviour of the empirical series analysed in Section 5. We restrict the set of ϕ_1 values to the range $0.1 \leqslant \phi_1 \leqslant 0.4$. Values of $\phi_1 < 0$ are often less encountered in practice while values of $\phi_1 \geqslant 0.5$ represent such a bad identification which would be found by an inspection of the residual, and squared residual autocorrelations. To make the rejection rates comparable across statistics, the estimated rejection rates are size-adjusted, i.e. the size being common for all three portmanteau tests.

Figure 2 about here

Figure 2 shows size-adjusted, empirical power results for $n = 100$, $\phi_1 = 0.2$, and 0.4, at the nominal level of 0.05. Overall, the results in Figure 2 suggests that $n^{-2}\hat{Q}_m$ has good power as opposed to the Q_m^{LB} and D_m statistics. In particular, when $m - p^*$ is small $n^{-2}\hat{Q}_m$ appears to work well. It is interesting to see that this kind of simple model misspecification is not easily found by the traditional Q_m^{LB} statistic and the new D_m statistic.

5 Example

We now illustrate our test on a time series X_t of 369 IBM daily closing stock prices, covering the period 17th May 1961–2nd November 1962. The series is listed in Box and Jenkins (1976) (Series B). They fitted an ARIMA(0,1,1) model $X_t - X_{t-1} = \varepsilon_t + 0.09\varepsilon_{t-1}$ to the raw series, but find evidence of model inadequacy. The inadequacy may be due to a change in the MA parameter, i.e. a change in the autocorrelation structure. Wichern, Miller and Hsu (1976) adapted an AR(1) model for the first differences $Y_t = \log(X_t) - \log(X_{t-1})$. Figure 3.a) shows a plot of $\{Y_t\}$. Subsequently, these authors identified two potential time points for a step change in the variance, at data points 180 and 235.

Figure 3 about here

Later, Tyssedal and Tjøstheim (1988) reanalysed the $\{Y_t\}$ series. They determined a change point in the second week of May 1962 (at data point 248). Going back to some historical records Tyssedal and Tjøstheim noted that the Dow Jones index dropped sharply in May 1962, with on May 28th its largest fall ever recorded since the 1929 crash. This fall is attributed to a conflict between the Kennedy Administration and several leading steel corporations. Early April 1962 United States Steel announced a price increase averaging \$6.00 a ton, and the other large steel companies were expected to follow suit. In his public statements President Kennedy escalated the steel controversy into a crisis; he made it known, through word and symbolic deeds, that his administration would not stand for the steel companies' behavior. Figure 3.b) presents plots of $\hat{R}(j,k) = \sum_{t=k+1}^{j} Y_t Y_{t-k}$ against j ($j = k+1, k+2, \ldots, 368; k = 1, 2$ and 3). For all values of k we see changing patterns of $\hat{R}(j,k)$ between $j = 230$ and $j = 250$, confirming Tyssedal and Tjøstheim's (1988) conclusion.

Table 2 about here

As a start, values of the test statistics Q_m^{LB} and D_m ($m = 2, \ldots, 33$) were computed, using residuals obtained from an AR(1) model fitted to $\{Y_t\}$. For all selected values of m the test statistic Q_m^{LB} rejects the H_0 of white noise at the 5% level, while this is not the case with the test statistic D_m. Next, to allow for a fair comparison between the test statistics $n^{-2}\hat{Q}_m$, Q_m^{LB}, and D_m, we fitted AR(p^*) models to the series $\{Y_t\}$ with $p^* = 5, 6, \ldots, 27$. In fact, using AIC, the "best" model identified in the class of pure AR models is an AR(27).

Table 2 provides summary information on the significance of the three test statistics. In the columns labeled $p^* = 5, 6, \ldots, 11$, three entries are given: The first entry concerns test results of $n^{-2}\hat{Q}_m$, the second entry test results of Q_m^{LB}, and the third entry test results of D_m. A "$+$" denotes significantly different from zero at the 5% level, a "$-$" denotes not significantly different from zero at the 5% level; and "$\times$" denotes no results for the D_m test are available for a particular combination of m and p^* due to the fact that $\alpha \leq 0$ or $\beta \leq 0$; see also footnote 2. In the columns labeled $p^* = 12, 13, \ldots, 27$, test information is only provided for $n^{-2}\hat{Q}_m$ and Q_m^{LB} (2 entries per column). The asymptotic 5% critical values of $n^{-2}\hat{Q}_m$ were computed by the $a\chi^2_{b-p^*}$ approximation.

From Table 2 we make the following observations. When $p^* = 5$, $n^{-2}\hat{Q}_m$ rejects the H_0 of no residual autocorrelation for $m = 6$, 7, and 8. Both Q_m^{LB} and D_m display similar results for the entire range of possible values m. When $6 \leq p^* \leq 17$, there is no indication to reject the H_0 on the basis of the $n^{-2}\hat{Q}_m$ test statistic. On the other hand, when $6 \leq p^* \leq 15$, Q_m^{LB} detects residual autocorrelation for almost all values of m. The D_m statistic reveals similar results for $p^* \leq 11$. The lack of model adequacy detected by these last two test statistics is mainly due to a single significant residual autocorrelation at lag 16.

When $18 \leq p^* \leq 27$, similar conclusions about H_0 emerge from both test statistics $n^{-2}\hat{Q}_m$ and Q_m^{LB} for several combinations of m and p^*. Particularly, the results for $(m - p^*) \leq 4$ give ground to reject the hypothesis of model adequacy at the 5% level. Note, however, that for $p^* = 26$ and 27, $n^{-2}\hat{Q}_m$ still provides an indication of model misspecification while this is no longer the case with Q_m^{LB}. Hence, it is recommended not to rely completely on the results of the Q_m^{LB} statistic, with the obvious caveat that our results for $n^{-2}\hat{Q}_m$ are intended as an illustration of the use of the proposed test in practice, not as an in-depth analysis of the series under study.

6 Concluding remarks

A new statistic $n^{-2}\hat{Q}_m$ for testing white noise has been proposed. The test is based on partial sums of lagged cross-products of AR residuals $\hat{R}(j,k)$. These latter quantities may be viewed as useful complements to the partial sums of squared residual process $\hat{R}(j,0)$, often used for detecting parameter changes. Indeed, we showed in Section 5 that $n^{-2}\hat{Q}_m$ can aid in determining changes in (residual) correlation structure in conjunction with an examination of the sample paths of $\hat{R}(j,k)$. Moreover, our simulation study indicated that $n^{-2}\hat{Q}_m$ has satisfactory null distribution and power, as compared to two other portmanteau tests, if $m-p^*$ is properly chosen. Thus, the proposed test can complement other tests when there is structural change in the autocorrelations.

Various additional simulations may be considered, including testing the adequacy of nonlinear time series models using squared residuals. This will be an objective of further study. From a theoretical point of view, we believe that asymptotic results on the higher moments of partial sums of lagged cross-products of residuals, i.e. $\sum_{t=k+1}^{j}(\hat{\varepsilon}_t\hat{\varepsilon}_{t-k})^i$ $(i=2,3,\ldots)$, are needed. Based on these results new (robust) goodness-of-fit tests for (non)linear time series processes with heavy tailed innovations can be developed.

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Table 1: Some selected quantiles of the test statistic $n^{-2}\hat{Q}_m$ for the white noise process $\{\varepsilon_t\}$.

	Probability								
m	0.01	0.025	0.05	0.1	0.5	0.90	0.95	0.975	0.99
1	0.0345	0.0444	0.0565	0.0765	0.2905	1.1958	1.6557	2.1347	2.7875
2	0.1269	0.1603	0.1991	0.2603	0.7575	2.0622	2.6241	3.1859	3.9286
3	0.2645	0.3282	0.3998	0.5082	1.2480	2.8256	3.4596	4.0813	4.8907
4	0.4376	0.5350	0.6412	0.7964	1.7438	3.5410	4.2339	4.9054	5.7703
5	0.6393	0.7714	0.9119	1.1115	2.2414	4.2274	4.9715	5.6863	6.6005
6	0.8642	1.0308	1.2045	1.4458	2.7399	4.8937	5.6840	6.4384	7.3965
7	1.1081	1.3085	1.5139	1.7944	3.2389	5.5457	6.3784	7.1681	8.1664
8	1.3678	1.6011	1.8368	2.1542	3.7382	6.1859	7.0565	7.8802	8.9219
9	1.6411	1.9060	2.1707	2.5229	4.2377	6.8172	7.7256	8.5797	9.6343
10	1.9261	2.2214	2.5138	2.8994	4.7371	7.4418	8.3847	9.2752	10.4964
15	3.4783	3.9126	4.3288	4.8613	7.2358	10.4867	11.6476	12.5856	13.8400
20	5.1761	5.7311	6.2536	6.9106	9.7352	13.4313	14.6504	15.7716	17.1512
25	6.9582	7.6310	8.2478	9.0150	12.2348	16.3271	17.6573	18.8744	20.3647
30	8.8277	9.5888	10.2912	11.1581	14.7345	19.1840	20.6147	21.9194	23.5027

Table 2: Test results of $n^{-2}\hat{Q}_m$ (1st entry), Q_m^{LB} (2nd entry), and D_m (3rd entry) for the residuals of the AR(p^*) models fitted to the series Y_t: "+" denotes significantly different from zero at the 5% level; "−" denotes not significantly different from zero at the 5% level; "×" denotes no results for the D_m test are available due to the fact that $\alpha \leq 0$ or $\beta \leq 0$ for the gamma distribution approximation.

	p^*																						
m	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
6	+ − ×																						
7	+ − ×	− − ×																					
8	+ − ×	− − ×	− − ×																				
9	− − ×	− − ×	− − ×	− − ×																			
10	− − ×	− − ×	− − ×	− − ×	− − ×																		
11	− − ×	− − ×	− − ×	− − ×	− − ×	− − ×																	
12	− − ×	− − ×	− − ×	− − ×	− − ×	− − ×	− − ×																
13	− + ×	− − ×	− − ×	− − ×	− − ×	− − ×	− − ×	− −															
14	− + +	− − ×	− − ×	− − ×	− − ×	− − ×	− − ×	− −	− −														
15	− + +	− − ×	− − ×	− − ×	− − ×	− − ×	− − ×	− −	− −	− −													
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29	− + +	− + +	− + +	− + +	− + +	− + +	− + ×	− +	− +	− +	− +	− −	− −	− −	− −	− −	− −	− −	− −	− −	+ +	+ −	+ −
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32	− + +	− + +	− + +	− + +	− + +	− + +	− + +	− +	− +	− +	− +	− −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− −	+ −
33	− + +	− + +	− + +	− + +	− + +	− + +	− + +	− +	− +	− +	− +	− −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− −

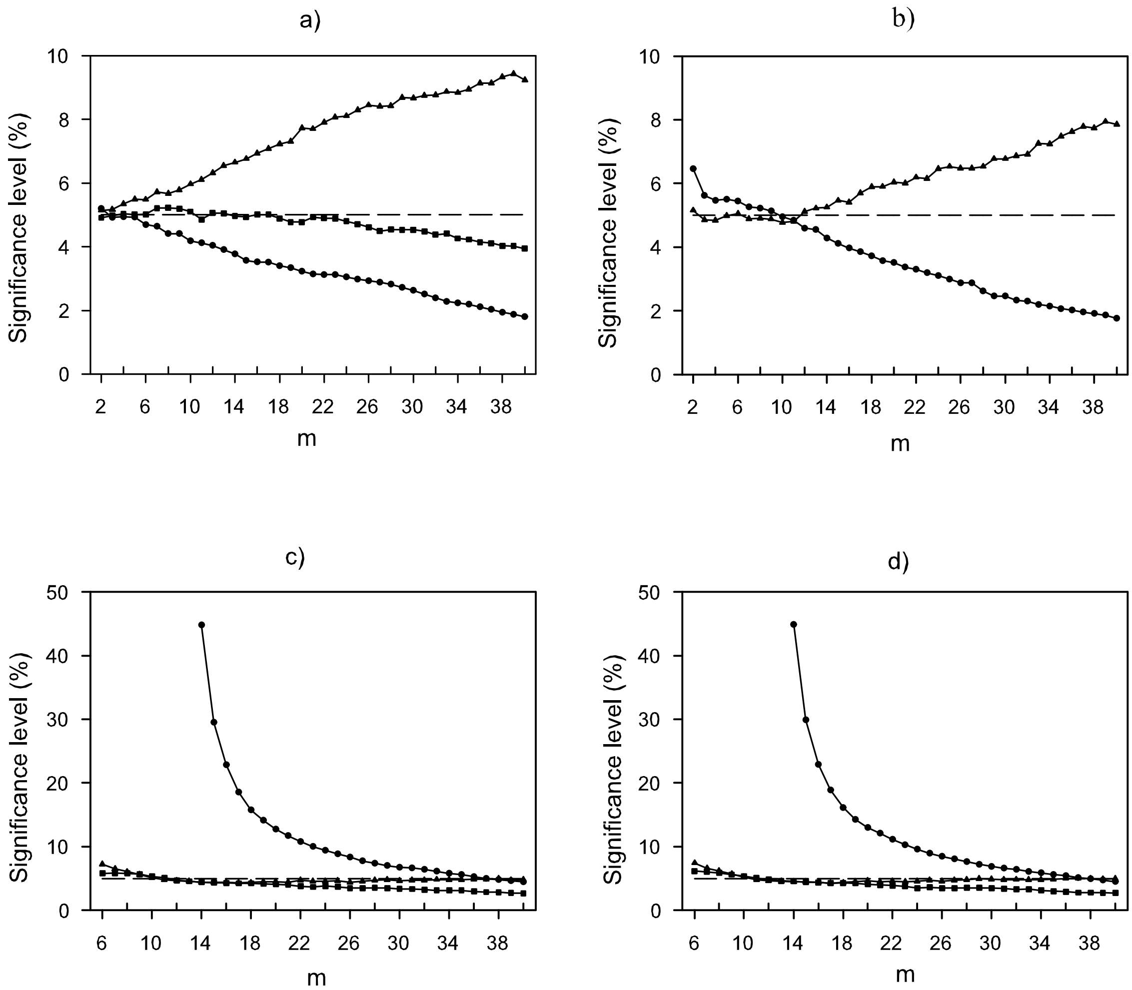

Figure 1: *Empirical significance levels of* $n^{-2}\hat{Q}_m$ *(squares),* Q_m^{LB} *(triangles), and* D_m *(circles),* $n = 100$*, 5% nominal significance level; a) white noise, b) true model AR(1) with* $\phi_1 = 0.3$ *and fitted model AR(1); c) true model AR(1) with* $\phi_1 = 0.3$ *and fitted model AR(5); d) true model AR(1) with* $\phi_1 = 0.5$ *and fitted model AR(5).*

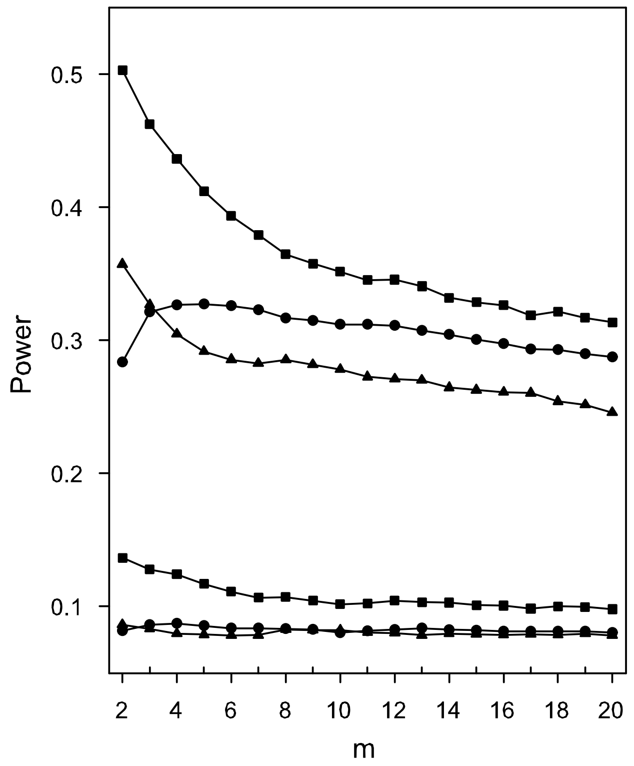

Figure 2: *Empirical powers of* $n^{-2}\hat{Q}_m$ *(squares),* Q_m^{LB} *(triangles), and* D_m *(circles) for* $X_t = \phi_1 X_{t-1} + \varepsilon_t$ *if* $t = 1, \ldots, n/2$ *and* $X_t = -\phi_1 X_{t-1} + \varepsilon_t$ *if* $t = n/2+1, \ldots, n$ *with* $\phi_1 = 0.2$ *(bottom three lines) and* $\phi_1 = 0.4$ *(top three lines);* $n = 100$.

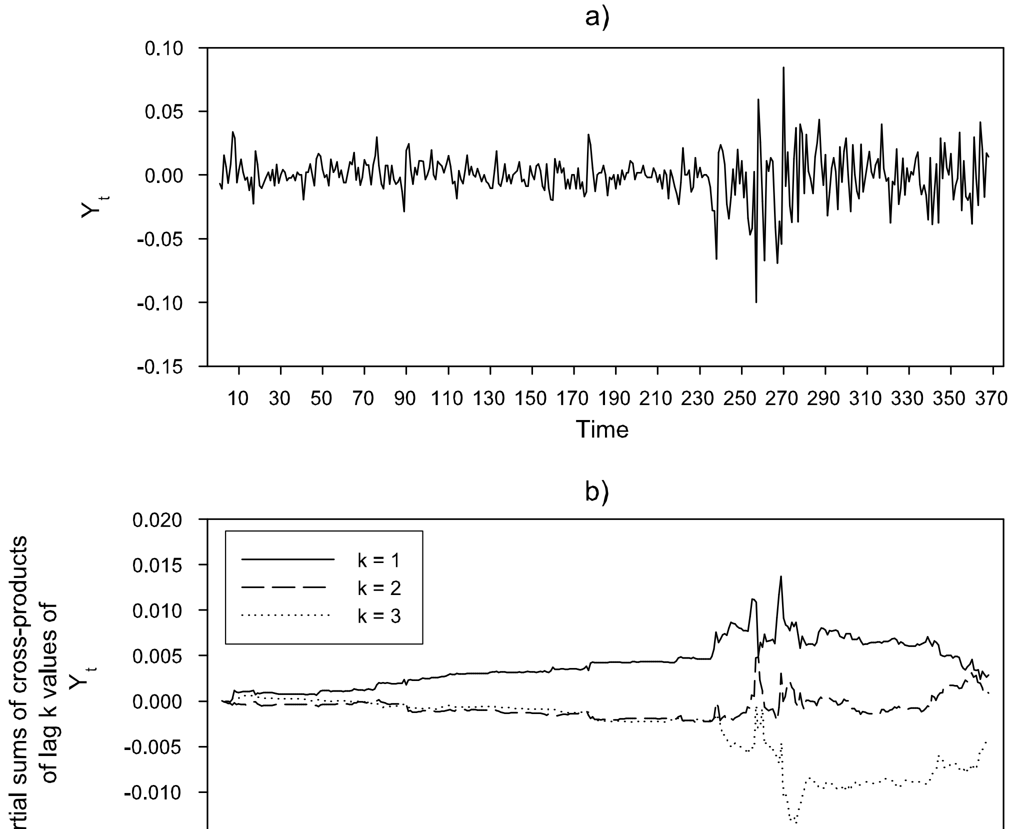

Figure 3: *a) Plot of* $Y_t = \log(X_t) - \log(X_{t-1})$*, for the IBM stock price data; b) Plots of* $\hat{R}(j,k) = \sum_{t=k+1}^{j} Y_t Y_{t-k}$ *against* j $(j = k+1, k+2, \ldots, 368; k = 1, 2, 3)$.