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Power Properties of Invariant Tests for Spatial Autocorrelation in Linear Regression

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Abstract: Many popular tests for residual spatial autocorrelation in the context of the linear regression model belong to the class of invariant tests. This paper derives some exact properties of the power function of such tests. In particular, we characterize the circumstances under which the limiting power, as the autocorrelation increases, vanishes, thus extending the work of Krämer (2005, Journal of Statistical Planning and Inference 128, 489-496). More generally, the analysis in the paper sheds new light on how the power of invariant tests for spatial autocorrelation is affected by the matrix of regressors and by the spatial structure. A numerical study aimed at assessing the practical relevance of the theoretical results is included.

Keywords: Cliff-Ord test; invariant tests; linear regression model; point optimal tests; power; similar tests; spatial autocorrelation.

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1 Introduction

Testing for residual spatial autocorrelation in the context of the linear regression model (e.g., Cliff and Ord, 1981, Anselin, 1988, Cressie, 1993) is now recognized as a crucial step in much empirical work in economics, geography and regional science. The present paper is concerned with finite sample power properties of tests used for this purpose. More specifically, we aim to shed some light on how power is affected by the regressors and by the spatial structure.

So far, power properties of tests for residual spatial autocorrelation have received much less attention than those of tests for residual serial correlation, and have mainly been studied by Monte Carlo simulation (see Florax and de Graaff, 2004, and references therein). Very few attempts have been made to derive exact properties of such tests, two notable exceptions being King (1981) and Krämer (2005). The former paper has established that the most popular test for spatial autocorrelation in regression residuals, the Cliff-Ord test, is locally best invariant for an important class of alternatives. The latter paper has generalized some results previously available for tests of serial autocorrelation (see Krämer, 1985, and Zeisel, 1989); in particular, Krämer (2005) has shown analytically that there are cases in which the power of some tests for spatial autocorrelation (namely, those whose associated test statistics can be expressed as ratios of quadratic forms in the regression residuals) can vanish as the spatial autocorrelation in the data increases. In general, it is fair to state that, while there is some evidence in the literature that the properties of tests for spatial autocorrelation can be very sensitive to the regressors and to the spatial structure, little is known about which combinations of regressors and spatial structures lead to low or high power.

Of course—given the popularity of the linear regression model and the pervasiveness of the issue of spatial autocorrelation in many empirical investigations—a large number of procedures are available for the purpose of testing for residual spatial autocorrelation, and one can choose among them on the basis of the suspected form of autocorrelation or on the basis of the desired properties of the testing procedure. In this paper, we confine ourselves to a rather simple, and extremely popular, framework. We assume that the regression errors follow a (first-order) *conditional or simultaneous autoregression* (e.g., Cressie, 1993) and we focus on *invariant tests* (e.g., Lehmann and Romano, 2005). Even in this simple setup the analytical investigation of exact power properties of tests is complicated. Because of the availability of many approximating techniques for power functions, this is not necessarily a problem when interest lies in the properties of a test in the context of a given model, i.e., when both the spatial structure and the matrix of regressors are fixed. However, when interest is, as in this paper, in how the properties of a test depend on the regressors and on the spatial structure, none of the available numerical or analytical approximations is likely to yield conclusive results. One feature of our approach is that some new properties of the power function of invariant tests are deduced directly from the density of the pertinent *maximal invariant* avoiding the need for complicated expressions for power functions, or approximations to them.

Throughout the paper, Gaussianity is assumed in order to treat conditional or simultaneous autoregressions in the same framework: indeed, while simultaneous autoregressions can be easily extended to non-Gaussian contexts, conditional autoregressions are inherently Gaussian (see Besag, 1974). It should be noted, however, that, since the density of the maximal invariant remains the same for any elliptically symmetric model (e.g., Kariya, 1980), the results that we obtain hold more generally for any SAR model based on an elliptically symmetric distribution.

Our results are as follows. Firstly, we extend the results of Krämer (2005) in several directions:

we formulate conditions for the limiting, as the autocorrelation increases, power of any invariant test to be 0, 1, or in $(0,1)$; we prove that, for any given spatial structure and irrespective of the size of the tests, there exists a set (of positive measure) of subspaces spanned by the regressors such that the limiting power of a locally best or point optimal invariant test vanishes; we characterize such "hostile" subspaces. Secondly, we discuss some conditions that are sufficient for unbiasedness of invariant tests for spatial autocorrelation and for monotonicity of their power function. Although such conditions are not necessary, they help to understand the causes of undesirable properties of the tests.

These results call for caution in interpreting the outcome of tests for residual spatial autocorrelation. In some circumstances, especially when the number of degrees of freedom is low and large autocorrelation is suspected, it may be very difficult to detect autocorrelation in the context of spatial autoregressions. Our results are also relevant outside a formal hypothesis testing framework, because they imply that there are circumstances in which the practice (e.g., Cliff and Ord, 1981, Anselin, 1988) of interpreting the Cliff-Ord statistic as an autocorrelation coefficient cannot be justified.

The theoretical framework in which we work is presented in Section 2. Section 3 analyzes the limiting power of invariant tests for spatial autocorrelation. This is done by first considering the general case of a model with arbitrary regressors, and then specializing the results to zero-mean models. A small numerical study of the practical relevance of the results is included. Further insights into the role played by the regressors and the spatial structure in determining the power of invariant tests of autocorrelation are gained in Section 4, by analyzing the conditions for unbiasedness of the tests and monotonicity of their power functions. Section 5 concludes by discussing some of limitations and possible extensions of our analysis. The Appendices contain technical material and all proofs.

2 The Setup

This section presents the context in which our results will be derived. Section 2.1 defines the testing problem, which is one of testing autocorrelation in a linear regression model. Section 2.2 briefly reviews the definition of invariant tests, and discusses the advantages of such tests for our testing problem.

2.1 The Testing Problem

Consider a *fixed* and *finite* set of n observational units, such as the regions of a country, and let $\boldsymbol{y} = (y_1, ..., y_n)'$, where y_i denotes the random variable observed at the i-th (according to some arbitrary ordering) unit. We assume that $\boldsymbol{y}$ is modelled according to a Gaussian linear regression model, i.e.,

$$\boldsymbol{y} \sim \mathrm{N}(\boldsymbol{X\beta}, \sigma^2 \boldsymbol{\Sigma}(\rho)), \tag{1}$$

where $\boldsymbol{X}$ is a non-stochastic $n \times k$ matrix of rank $k < n$, $\boldsymbol{\beta}$ is a $k \times 1$ vector of unknown parameters, σ^2 is an unknown positive parameter, and ρ is an unknown parameter belonging to some connected subset, to be denoted by Ψ, of the set of values of ρ such that $\boldsymbol{\Sigma}(\rho)$ is positive definite. We assume that the function $\rho \to \boldsymbol{\Sigma}(\rho)$ is differentiable, and that the parameters of the model are identified (in the sense that the parameter space of the model does not contain two distinct points indexing the same distribution) and functionally independent. Throughout the paper we will often refer to the case of a general $\boldsymbol{\Sigma}(\rho)$, but will be mostly concerned with the specific covariance structures implied by spatial autoregressive processes.

There are two distinct classes of Gaussian spatial autoregressions: conditional autoregressive (CAR) processes and simultaneous autoregressive (SAR) processes. They are both discussed extensively in many books and articles in the statistics and econometrics literature (e.g., Whittle, 1954, Besag, 1974, Cliff and Ord, 1981, Anselin, 1988, Cressie, 1993), to which we refer for details concerning the construction and interpretation of the models. Here, we only present the covariance matrices implied by the models. As in most of the theoretical and empirical literature on spatial autoregressive processes, we confine ourselves to *first-order* (or one-parameter) processes.[1] Such processes are specified on the basis of a fixed $n \times n$ (spatial) *weights matrix* $\boldsymbol{W}$, chosen to reflect a priori information on relations among the n observations. Typically, for each $i, j = 1, ..., n$, $W_{i,j} = 0$ if i and j are not neighbors according to some metric deemed to be relevant for the phenomenon under analysis, whereas $W_{i,j}$ is set to some non-zero number, possibly reflecting the degree of interaction, otherwise. For instance, if the observational units are the regions of a country, one may set $W_{i,j} = 1$ if two distinct regions i and j share a common boundary, $W_{i,j} = 0$ otherwise. In this paper we assume that a weights matrix (i) has zero entries along its main diagonal, (ii) is entrywise nonnegative, (iii) is irreducible. Details concerning such assumptions are in Appendix A.

Let $\boldsymbol{I}$ denote the $n \times n$ identity matrix. A CAR specification yields (see Besag, 1974)

$$\boldsymbol{\Sigma}(\rho) = (\boldsymbol{I} - \rho \boldsymbol{W})^{-1} \boldsymbol{L}, \tag{2}$$

where $\boldsymbol{L}$ is a fixed $n \times n$ diagonal matrix such that $\boldsymbol{L}^{-1}\boldsymbol{W}$ is symmetric. For each $i = 1, ..., n$, the term $\sigma^2 L_{i,i}$ represents the variance of y_i conditional on all the remaining random variables in $\boldsymbol{y}$. We remark that structure (2) constitutes a very natural framework for the study of autocorrelation tests, even without referring to a CAR interpretation; see, e.g., Anderson (1948), Kadiyala (1970), Kariya (1980), King (1980).

On the other hand, a SAR process implies

$$\boldsymbol{\Sigma}(\rho) = (\boldsymbol{I} - \rho \boldsymbol{W})^{-1} \boldsymbol{V} (\boldsymbol{I} - \rho \boldsymbol{W}')^{-1}, \tag{3}$$

where $\boldsymbol{V}$ is a fixed $n \times n$ symmetric and positive definite matrix. We note immediately that because of the assumptions made above that $\boldsymbol{X}$ is fixed and that $\boldsymbol{\beta}$ and ρ are functionally independent, models containing a spatial lag of $\boldsymbol{y}$ among the regressors are not in the class of models considered in this paper. We shall come back to this point in Section 5.

For the purposes of this paper, specifications (2) and (3) can be slightly simplified. This is because for our testing problem (defined in the next paragraph), there is no loss of generality in assuming that $\boldsymbol{\Sigma}(0) = \boldsymbol{I}$. Accordingly, from now on and unless otherwise specified, the term "CAR model" refers to the family of distributions

$$\mathrm{N}(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2 (\boldsymbol{I} - \rho \boldsymbol{W})^{-1}), \tag{4}$$

(for a fixed $\boldsymbol{W}$) and the term "SAR model" to the family

$$\mathrm{N}(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2 \left[(\boldsymbol{I} - \rho \boldsymbol{W}')(\boldsymbol{I} - \rho \boldsymbol{W}) \right]^{-1}), \tag{5}$$

(again, for a fixed $\boldsymbol{W}$). The normalization to $\boldsymbol{\Sigma}(0) = \boldsymbol{I}$ emphasizes a crucial difference between CAR and SAR models: in CAR models there is no loss of generality (as far as our purposes are concerned)

[1] First-order spatial autoregressions are usually denoted by CAR(1) and SAR(1). Throughout, we drop the specification of the order, since this paper is concerned only with first-order processes.

in assuming that $\boldsymbol{W}$ is a symmetric matrix, whereas in SAR models we have to allow explicitly the possibility of a nonsymmetric $\boldsymbol{W}$. In fact, we shall find that there are interesting differences between SAR models with a symmetric weights matrix, henceforth referred to as *symmetric SAR models*, and SAR models with a nonsymmetric weights matrix—henceforth referred to as *asymmetric SAR models.*

In the context of model (1), our testing problem consists of:

$$\mathrm{H}_0 : \rho = 0 \qquad \text{versus} \qquad \mathrm{H}_a : \rho > 0. \tag{6}$$

Here and throughout, $\rho > 0$ is to be understood as $\rho \in \mathbb{R}^+ \cap \Psi$, that is, we leave it implicit that ρ must belong to the parameter space of the model. The choice of a one-sided alternative rather than a two-sided one is dictated by the fact that the former is more relevant for many specifications of $\boldsymbol{\Sigma}(\rho)$. In particular, under the assumption introduced below, for both CAR and SAR models $\rho > 0$ represents positive spatial autocorrelation, a much more common phenomenon in practice than negative spatial autocorrelation. The assumption is that $\mathbb{R}^+ \cap \Psi = [0, \lambda_{\max}^{-1})$, where $\lambda_{\max}$ is the largest eigenvalue in modulus of $\boldsymbol{W}$. Some properties of $\lambda_{\max}$ that are important for the results obtained in this paper are pointed out in Appendix A. Here, we remark that the value $\rho = \lambda_{\max}^{-1}$ can be interpreted as the analog of a *unit root* in an AR(1) model (see, e.g., Fingleton, 1999, and Paulauskas, 2006).

2.2 The Tests

For the testing problem defined above, we will focus on the so-called *invariant* tests. These are now briefly introduced, with particular attention to point optimal invariant tests and locally best invariant tests; details on the theory of invariant tests are available in standard references such as Lehmann and Romano (2005). Generally speaking, a test is said to be invariant if it preserves the symmetries satisfied by the associated testing problem. More precisely, a test is said to be invariant with respect to a certain group of transformations of the sample space if it is based on a test statistic that is constant on each orbit of that group. A necessary and sufficient condition for this type of invariance is that the test statistic is a function of a *maximal invariant* under that group.

The testing problem (6) is invariant with respect to the group of transformations $\boldsymbol{y} \rightarrow a\boldsymbol{y} + \boldsymbol{X}\boldsymbol{b}$, with $a \in \mathbb{R}\backslash\{0\}$ and $\boldsymbol{b} \in \mathbb{R}^k$ (see Appendix B for more details). Let $\boldsymbol{C}$ be an $(n-k) \times n$ matrix such that $\boldsymbol{C}\boldsymbol{C}'$ equals the identity matrix of order $n-k$ and $\boldsymbol{C}'\boldsymbol{C} = \boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' =: \boldsymbol{M}_{\boldsymbol{X}}$, and let $\|\cdot\|$ denote the Euclidean norm. Fix, without any loss of generality, an arbitrary $\bar{\imath} = 1, ..., n$. Then, a maximal invariant under the above group is $\boldsymbol{v} := \operatorname{sgn}(y_{\bar{\imath}})\boldsymbol{C}\boldsymbol{y}/\left\|\boldsymbol{C}\boldsymbol{y}\right\|$. Its density, with respect to the normalized Haar measure on the hemisphere $S_{n-k} := \left\{\boldsymbol{s} \in \mathbb{R}^{n-k} : \|\boldsymbol{s}\| = 1, s_{\bar{\imath}} \geq 0\right\}$, is

$$pdf(\boldsymbol{v}; \rho) = 2\left|\boldsymbol{C}\boldsymbol{\Sigma}(\rho)\boldsymbol{C}'\right|^{-\frac{1}{2}} \left[\boldsymbol{v}' \left(\boldsymbol{C}\boldsymbol{\Sigma}(\rho)\boldsymbol{C}'\right)^{-1} \boldsymbol{v}\right]^{-\frac{n-k}{2}} \tag{7}$$

(see Kariya, 1980, equation (3.7)).

Besides the "principle of invariance", there are at least two other reasons why invariant tests are particularly appealing for our testing problem. Firstly, invariant tests can be implemented easily. Since an invariant test statistic must depend on $\boldsymbol{y}$ only through $\boldsymbol{v}$, its distribution under the null (and also under the alternative) is free of nuisance parameters, and hence critical values can, in general, be obtained accurately by Monte Carlo or other numerical methods. Secondly, expression (7) turns out to be proportional to the *marginal likelihood* of ρ, which has often been found to provide a better basis for inference about ρ than the full likelihood of model (1) (especially when k is large with respect to n); see, for instance, Tunnicliffe Wilson (1989) and Rahman and King (1997).

It is well known that in general, despite the elimination of the nuisance parameters achieved in (7), no uniformly most powerful invariant (UMPI) test exists for the testing problem (6). One could then resort to a test that is optimal according to some exact criterion (see, for instance, Cox and Hinkley, 1974, p. 102), or to a test that has less clear-cut optimality properties but performs well in general, such as a likelihood ratio test (which is an invariant test; e.g. Cox and Hinkley, 1974, p. 173) or its restricted version based on $pdf(\boldsymbol{v};\rho)$. In the present paper we are particularly concerned with the tests—named *point optimal invariant* (POI) tests by King (1988)—that are the most powerful amongst all invariant tests against a specific alternative $\rho = \bar{\rho} > 0$, and with the *locally best invariant* (LBI) test, which is obtained as the limiting case for $\bar{\rho} \to 0$. In general, and certainly for our testing problem, the locally most powerful test coincides with the test maximizing the slope of the power function at $\rho = 0$ (see Lehmann and Romano, 2005, p. 339). The main reasons why we focus on POI and LBI tests are: (i) POI tests define the upper bound—the so-called power envelope—to the power attainable by any invariant test of a fixed size; (ii) LBI tests are extremely popular in practice, especially in the context of CAR and SAR models (see below).

Next, we define the critical regions associated to POI and LBI tests for our testing problem. The size of a critical region is denoted by α and, to avoid trivial cases and unless otherwise specified, is assumed to be in $(0,1)$. Application of the Neyman-Pearson Lemma to the density (7) reveals that the critical region of a POI test at the point $\bar{\rho}$ is defined by

$$\boldsymbol{v}' \left(\boldsymbol{C}\boldsymbol{\Sigma}(\bar{\rho})\boldsymbol{C}'\right)^{-1} \boldsymbol{v} < c_\alpha, \tag{8}$$

where c_α is a constant such that the size of the test is α.[2] Denoting by $\pi_{\bar{\rho}}(\rho)$ the power of such a critical region, the power envelope of size-α invariant tests is the function that associates the value $\pi_\rho(\rho)$ to each $\rho \geq 0$. When needed, we will emphasize the dependence of $\pi_\rho(\rho)$ on $\boldsymbol{X}$ by writing $\pi_\rho(\rho;\boldsymbol{X})$. For a fixed size α, the LBI critical region is defined by

$$\boldsymbol{v}'\boldsymbol{C}\boldsymbol{A}_0\boldsymbol{C}'\boldsymbol{v} < c_\alpha, \tag{9}$$

where $\boldsymbol{A}_0 = d\boldsymbol{\Sigma}^{-1}(\rho)/d\rho\big|_{\rho=0}$ (King and Hillier, 1985). When $-\boldsymbol{A}_0$ is equal to some spatial weights matrix $\boldsymbol{W}$ (or to $\boldsymbol{W}+\boldsymbol{W}'$), as it is in the case of CAR or SAR models, the LBI test is known in the literature as Cliff-Ord test (see Cliff and Ord, 1981, King, 1981, Kelejian and Prucha, 2001). The Cliff-Ord test represents the generalization to regression residuals of the Moran test (Moran, 1950), and is, by far, the most popular test for spatial autocorrelation in regression models.

Before we continue, some notation is in order. For a $q \times q$ symmetric matrix $\boldsymbol{Q}$, we denote by $\lambda_1(\boldsymbol{Q}), ..., \lambda_q(\boldsymbol{Q})$ its eigenvalues, labeled in non-decreasing order of magnitude; by $m_i(\boldsymbol{Q})$ the multiplicity of $\lambda_i(\boldsymbol{Q})$, for $i = 1, ..., q$; by $\boldsymbol{f}_1(\boldsymbol{Q}), ..., \boldsymbol{f}_q(\boldsymbol{Q})$ a set of orthonormal (with respect to the Euclidean norm) eigenvectors of $\boldsymbol{Q}$, with the eigenvector $\boldsymbol{f}_i(\boldsymbol{Q})$ being pertinent to the eigenvalue $\lambda_i(\boldsymbol{Q})$; by $E_i(\boldsymbol{Q})$ the eigenspace associated to $\lambda_i(\boldsymbol{Q})$, for $i = 1, ..., q$. Note that, when $\boldsymbol{W}$ is symmetric, $\lambda_n(\boldsymbol{W}) = \lambda_{\max}$. When $\boldsymbol{W}$ is nonsymmetric, its eigenvalues cannot in general be ordered as above, but $\lambda_{\max}$ is still well-defined.

[2] Note that in (8), $\boldsymbol{v}$ denotes a realization of the maximal invariant. Throughout the paper, we do not distinguish notationally between a random variable and its realizations.

3 Limiting Power

In Section 3.1 we discuss some preliminary results in the context of the general model (1). Then, for CAR and SAR models, we analyze the limiting power of invariant tests, as the autocorrelation increases. We consider the case with general regressors in Section 3.2, and without regressors in Section 3.3. Special attention is paid to the cases in which the limiting power vanishes, and to facilitate their analysis, we will refer to the minimum size such that the limiting power does not vanish. Such a value of the size of a critical region can be interpreted as a measure of difficulty of testing for autocorrelation. Finally, in Section 3.4 we report results from numerical experiments aimed at assessing the practical relevance of the theoretical results.

3.1 Preliminaries

We now present two results that set the scene for the analysis to follow. They both address the question of how the power of invariant tests of $\rho = 0$ versus $\rho > 0$ in model $\mathrm{N}(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{\Sigma}(\rho))$ is affected by $\boldsymbol{X}$, for some covariance structure $\boldsymbol{\Sigma}(\rho)$.

The first result concerns a comparison of the power envelope $\pi_\rho(\rho; \boldsymbol{X})$ when $\boldsymbol{X} \neq \mathbf{O}$ and when $\boldsymbol{X} = \mathbf{O}$. Henceforth, $\mathrm{col}(\boldsymbol{X})$ denotes the column space of the matrix $\boldsymbol{X}$.

Proposition 3.1 *Consider testing $\rho = 0$ versus $\rho > 0$ in model $\mathrm{N}(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{\Sigma}(\rho))$. For any $\boldsymbol{X} \neq \mathbf{O}$, any $\rho > 0$, and any α,*

$$\pi_\rho(\rho; \boldsymbol{X}) \leq \pi_\rho(\rho; \mathbf{O}). \tag{10}$$

In (10) equality is attained if and only if, for some $i = 2, ..., n-1$, $\mathrm{col}(\boldsymbol{X}) \subseteq E_i(\boldsymbol{\Sigma}(\rho))$ and $\alpha = \Pr(\boldsymbol{v}'\boldsymbol{\Sigma}^{-1}(\rho)\boldsymbol{v} < \lambda_i^{-1}(\boldsymbol{\Sigma}(\rho))$.

The circumstances leading to equality in (10) are extremely restrictive, because they pose severe constraints on $\boldsymbol{X}$, α, and $\boldsymbol{\Sigma}(\rho)$. Proposition 3.1 then asserts that, except for such restrictive circumstances, the presence of regressors has a detrimental effect (with respect to the case in which the model does not contain regressors, and as long as $\boldsymbol{\beta}$ is unknown) on the maximum power achievable by an invariant test for residual autocorrelation. Since it holds for any $\boldsymbol{\Sigma}(\rho)$, any $\boldsymbol{X}$, and any α, this is a rather general result on the role of regressors in testing for residual autocorrelation.

A brief discussion of some limitations of Proposition 3.1 is useful to motivate, and to highlight the difficulties of, the analysis to follow. First, note that inequality (10) involves two models (the model with $\boldsymbol{X} = \mathbf{O}$ and the model with some $\boldsymbol{X} \neq \mathbf{O}$) with different degrees of freedom. Secondly, it should be observed that in practice, one is usually more concerned with the power of a specific test than with a power envelope.[3] In light of these comments, an interesting question is which matrices $\boldsymbol{X}$ of a fixed dimension $n \times k$ are favorable, and which are less favorable, to the power of a particular invariant test, given $\boldsymbol{\Sigma}(\rho)$. This is a difficult question, because the answer in general *depends on ρ and α*. In the present paper, we will provide some answers concerned with specific cases: we will consider the

[3] For a general $\boldsymbol{\Sigma}(\rho)$, Proposition 3.1 does not imply that the power function of a particular invariant test when $\boldsymbol{X} = \mathbf{O}$ is uniformly (over $\rho > 0$) non-smaller than when $\boldsymbol{X} \neq \mathbf{O}$. Interestingly, such an implication does hold when $\boldsymbol{\Sigma}(\rho)$ is that of a CAR model and the test in question is POI or LBI, because for a zero-mean CAR model the POI critical region (8) does not depend on $\bar{\rho}$ (that is, there exists a UMPI test), and hence the power function of any POI or LBI test coincides with $\pi_\rho(\rho, \mathbf{O})$.

$\boldsymbol{\Sigma}(\rho)$ implied by a CAR or SAR model, we will mainly focus on large ρ, and, when necessary, we will restrict attention to POI and LBI tests.[4]

The second preliminary result presented here considers the limit of the power of an arbitrary invariant test as ρ approaches some positive value a (from the left, and with a an accumulation point of Ψ), and links such a limit to the limiting eigenstructure of $\boldsymbol{\Omega}_\rho := \boldsymbol{C}\boldsymbol{\Sigma}(\rho)\boldsymbol{C}'$. For notational convenience, let $\boldsymbol{\Omega} := \lim_{\rho \to a} \boldsymbol{\Omega}_\rho$.

Lemma 3.2 *Consider an arbitrary invariant critical region for testing $\rho = 0$ against $\rho > 0$ in model $\mathrm{N}(\boldsymbol{X\beta}, \sigma^2\boldsymbol{\Sigma}(\rho))$, and let a be some positive value such that $\boldsymbol{\Sigma}(\rho)$ is positive definite for $\rho \in (0, a)$ and for $\rho \to a$. When $\lambda_{n-k}(\boldsymbol{\Omega})$ is finite, the power of the critical region tends, as $\rho \to a$, to a number strictly between 0 and 1. When $\lambda_{n-k}(\boldsymbol{\Omega})$ is infinite and has algebraic multiplicity one, the power tends, as $\rho \to a$, to 1 if the critical region contains $\boldsymbol{f}_{n-k}(\boldsymbol{\Omega})$, to 0 otherwise.*

Lemma 3.2 holds for very general matrices $\boldsymbol{\Sigma}(\rho)$, including those of CAR, SAR, and (covariance stationary) AR(1) models. Clearly, the most alarming implication of Lemma 3.2 is that, for some combinations of $\boldsymbol{\Sigma}(\rho)$ and $\boldsymbol{X}$, power can vanish as $\rho \to a$. We now briefly comment on some intuitions underlying this phenomenon; details are in Appendix C. In the context of Lemma 3.2, power vanishes when the statistical model tends, as $\rho \to a$, to a family of distributions supported on a subset of the sample space $\mathbb{R}^n$, and such a subset does not intersect the critical region. Now, CAR and SAR models tend, as $\rho \to \lambda_{\max}^{-1}$, to be supported on a subspace of $\mathbb{R}^n$, for any $\boldsymbol{W}$, whereas an AR(1) model tends, as $\rho \to 1$, to be supported on a subspace of $\mathbb{R}^n$ only if the initial condition is such that the process is covariance stationary. Such observations have two important implications for the analysis to follow. Firstly, the zero power phenomenon is attributable to a characteristic of the statistical model, not of a particular test. Secondly, the phenomenon deserves particular attention in CAR and SAR models, because, as $\rho \to \lambda_{\max}^{-1}$, such models are always concentrated on a subspace of $\mathbb{R}^n$.

3.2 Main Results

We now restrict attention to the power of invariant tests in CAR and SAR models as $\rho \to \lambda_{\max}^{-1}$ (from the left). Values of ρ close to $\lambda_{\max}^{-1}$ are particularly relevant when the dependent variable $\boldsymbol{y}$ (given $\boldsymbol{X}$) exhibits a type of near nonstationarity similar to that due to a unit root in time-series autoregressions (see Fingleton, 1999, Mur and Trivez, 2003, Lauridsen, 2006).[5] For an empirical application involving a SAR model with ρ close to $\lambda_{\max}^{-1}$, see, for instance, Kosfeld and Lauridsen (2004). Consideration of the extreme case $\rho \to \lambda_{\max}^{-1}$ in the context of a regression model is interesting also because it corresponds, in general, to studying power when it is most needed, i.e., when the inefficiency of the OLS estimator of $\boldsymbol{\beta}$ is large (see Krämer and Donninger, 1987).

Henceforth, by "limiting power" we mean the limit of the power function as $\rho \to \lambda_{\max}^{-1}$. By $\Phi_{\boldsymbol{y}}$ we denote an invariant critical region defined on the sample space $\mathbb{R}^n$; later, we shall use $\Phi_{\boldsymbol{v}}$ for a critical region defined as a subset of S_{n-k}. Finally, $\boldsymbol{f}_{\max}$ denotes an eigenvector of $\boldsymbol{W}$ pertaining to $\lambda_{\max}$; see Appendix A for a precise definition. The key result of this section is the following theorem.

Theorem 3.3 *In CAR and SAR models, the limiting power of an invariant critical region $\Phi_{\boldsymbol{y}}$ for testing $\rho = 0$ against $\rho > 0$ is:*

[4] Other partial answers are available for the $\boldsymbol{\Sigma}(\rho)$ implied by an AR(1) process; e.g., Tillman (1975).

[5] The literature on spatial autoregressions with large ρ is particularly substantial outside economics; see, for instance, Besag and Kooperberg (1995) and Bhattacharyya *et al.* (1997), and references therein.

– *in* $(0, 1)$ *if* $\boldsymbol{f}_{\max} \in \mathrm{col}(\boldsymbol{X})$*;*

– 1 *if* $\boldsymbol{f}_{\max} \in \Phi_{\boldsymbol{y}} \backslash \mathrm{col}(\boldsymbol{X})$*;*

– 0 *otherwise.*

Theorem 3.3 asserts that, to some extent, the limiting power of $\Phi_{\boldsymbol{y}}$ is determined by which of three mutually disjoint subsets of the sample space ($\mathrm{col}(\boldsymbol{X})$, $\Phi_{\boldsymbol{y}} \backslash \mathrm{col}(\boldsymbol{X})$, and the complement of $\Phi_{\boldsymbol{y}} \cup \mathrm{col}(\boldsymbol{X})$) $\boldsymbol{f}_{\max}$ belongs to. This result is strongly related to Theorems 1 and 2 in Krämer (2005), the most important differences being: (i) the class of tests considered there (i.e., tests that can be expressed as ratios of quadratic forms in the regression residuals) and the class considered in the present paper (i.e., invariant tests) are different, although they intersect; (ii) our result does not require symmetry of $\boldsymbol{W}$.

It is worth stressing that Theorem 3.3 holds for *any* invariant critical region, regardless of the analytical form of the associated test statistic. In particular, it also holds for invariant tests whose test statistics are analytically complicated, or, as in the case of a likelihood ratio test,[6] unavailable in closed form.

The practical usefulness of Theorem 3.3 is in providing simple conditions for the limiting power of an invariant test to vanish, given any matrices $\boldsymbol{X}$ and $\boldsymbol{W}$. Consider an invariant critical region $\Phi_{\boldsymbol{y}}$ that rejects $\rho = 0$ for small values of some statistic $T(\boldsymbol{y})$, i.e.,

$$\Phi_{\boldsymbol{y}} = \{\boldsymbol{y} \in \mathbb{R}^n : T(\boldsymbol{y}) < c_\alpha\}. \tag{11}$$

Theorem 3.3 asserts that the limiting power of such a critical region is in $(0, 1)$ if $\boldsymbol{f}_{\max} \in \mathrm{col}(\boldsymbol{X})$, is 1 if $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$ and $T(\boldsymbol{f}_{\max}) < c_\alpha$, and is 0 if $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$ and $T(\boldsymbol{f}_{\max}) \geq c_\alpha$. Such conditions are typically simple to check because, in most cases, (i) $\boldsymbol{f}_{\max}$ is either known (e.g., it is a vector of identical entries when $\boldsymbol{W}$ is row-standardized; see Appendix A) or can be computed efficiently (e.g., by the power method); (ii) since $\Phi_{\boldsymbol{y}}$ is similar, c_α can be obtained accurately by simulation or other numerical methods. For instance, it is readily verified that, for the $\boldsymbol{\Sigma}(\rho)$ implied by CAR or SAR models, the limiting power of the LBI test is in $(0, 1)$, 1, or 0, depending on whether

$$\boldsymbol{f}'_{\max} (\boldsymbol{M}_{\boldsymbol{X}} \boldsymbol{A}_0 \boldsymbol{M}_{\boldsymbol{X}} - c_\alpha \boldsymbol{M}_{\boldsymbol{X}}) \boldsymbol{f}_{\max} \tag{12}$$

is respectively zero, negative, or positive. Analogously, simple algebra reveals that the limiting power of a POI test is in $(0, 1)$, 1, or 0, depending on whether

$$\boldsymbol{f}'_{\max} \left\{ \boldsymbol{\Sigma}^{-1}(\bar{\rho}) \left[\boldsymbol{I} - \boldsymbol{X} (\boldsymbol{X}' \boldsymbol{\Sigma}^{-1}(\bar{\rho}) \boldsymbol{X})^{-1} \boldsymbol{X}' \boldsymbol{\Sigma}^{-1}(\bar{\rho}) \right] - c_\alpha \boldsymbol{M}_{\boldsymbol{X}} \right\} \boldsymbol{f}_{\max} \tag{13}$$

is respectively zero, negative, or positive. Since they refer to test statistics that are ratios of quadratic forms in $\boldsymbol{y}$, the conditions based on (12) and (13) reduce, in the case of a symmetric SAR model, to conditions given in Krämer (2005).

In the rest of this subsection we take a close look at the vanishing of the limiting power, and, consequently, we restrict attention to the case $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$. Suppose that, by application of Theorem 3.3, one finds that, in the context of a given CAR or a SAR model, the limiting power of a certain critical region $\Phi_{\boldsymbol{y}}$ vanishes. Theorem 3.3 itself guarantees that if $\Phi_{\boldsymbol{y}}$ is enlarged so as to include $\boldsymbol{f}_{\max}$, then its limiting power jumps to 1. A practically important question is how large $\Phi_{\boldsymbol{y}}$

[6] Both a LR test based on the full likelihood and its version based on the marginal likelihood are unavailable in closed form, and, as mentioned in Section 2.2, belong to the class of invariant tests.

must be in order to avoid the vanishing of the limiting power. To explore this issue, it is convenient to refer to the minimum size such that the limiting power of a certain test does not vanish. More formally, we define (allowing, for convenience and contrary to what is done elsewhere in the paper, α to take the value 1):

Definition 3.4 *For an invariant test of $\rho = 0$ against $\rho > 0$ in a CAR or SAR model, α^* is the infimum of the set of values of $\alpha \in (0, 1]$ such that the limiting power does not vanish.*

When $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$, α^* is a measure of the distinguishability between the null hypothesis $\rho = 0$ and the alternative $\rho \to \lambda_{\max}^{-1}$. A large α^* indicates that a large critical region is necessary to avoid the zero limiting power problem; $\alpha^* = 0$ indicates that the limiting power is 1 for any α; finally, $\alpha^* = 1$ means that the limiting power is 0 for any α.[7] Note, in particular, that $\alpha^* = 1$ identifies the extremely negative situation in which the zero limiting power problem cannot be avoided by increasing α. A simple characterization of α^* is given in the following result.

Lemma 3.5 *When an invariant critical region for testing $\rho = 0$ against $\rho > 0$ in a CAR or SAR model is in form (11), and provided that $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$,*

$$\alpha^* = \Pr(T(\boldsymbol{y}) < T(\boldsymbol{f}_{\max});\ \boldsymbol{y} \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{I})). \tag{14}$$

The probability in (14) does not depend on any unknown parameter, and hence can be approximated accurately by simulation or other numerical methods. We stress that α^* depends on $\boldsymbol{X}$ (through $\mathrm{col}(\boldsymbol{X})$, because of the invariance property of the tests), $\boldsymbol{W}$, the invariant test under analysis, and on whether a CAR or a SAR model is considered. In particular, for a given $\boldsymbol{W}$, a given test, and a given k, α^* may depend to a very large extent on $\mathrm{col}(\boldsymbol{X})$. In Section 3.4 we will report some numerical examples, while Lemmata F.1 and G.1 give conditions for the case $\alpha^* = 0$. In the following, we focus on the other, and most worrisome, extreme, i.e., $\alpha^* = 1$. For simplicity, we restrict attention to POI and LBI tests.

Theorem 1 of Krämer (2005) contains the crucial statement that, in a symmetric SAR model, "given any matrix $\boldsymbol{W}$ of weights, and independently of sample size, there is always some regressor $\boldsymbol{X}$ such that for the Cliff–Ord test the limiting power disappears" (note that here "some regressor $\boldsymbol{X}$" refers to the case $k = 1$). Now, from Theorem 3.3 it is clear that whether or not a particular $\boldsymbol{X}$ (with $k \geq 1$) causes the limiting power to disappear depends on α. Thus, if interpreted as holding for any α, the above statement would imply that for any $\boldsymbol{W}$ there exists at least one regressor such that $\alpha^* = 1$ for the Cliff-Ord test (i.e., the LBI test) in a symmetric SAR model. Unfortunately, whether this implication is correct remains to be established, because the proof of Krämer's theorem holds only when $\alpha \to 0$.[8] The next result settles the issue and places it in a more general context. Recall that m_1 denotes the multiplicity of λ_1, for a symmetric $\boldsymbol{W}$. Generally $m_1 = 1$, unless $\boldsymbol{W}$ satisfies some symmetries; see, for instance, Biggs (1993).

[7] Recall that we are here focusing on the case $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$. If $\boldsymbol{f}_{\max} \in \mathrm{col}(\boldsymbol{X})$, α^* is always zero, by Theorem 3.3, and hence uninformative. In order to study the power of invariant tests when $\boldsymbol{f}_{\max} \in \mathrm{col}(\boldsymbol{X})$, one could define α^* as the infimum of the set of values such that the limiting power is greater than some positive value, but this is not pursued in the present paper.

[8] This is because d_1 in equation (12) of Krämer (2005) is not necessarily positive for any $\boldsymbol{W}$, unless $\alpha \to 0$. As a consequence, the regressors that Krämer constructs in his proof do not need to cause the limiting power to vanish when $d_1 < 0$.

Theorem 3.6 *Consider a POI or LBI test for $\rho = 0$ against $\rho > 0$ in the context of a CAR or symmetric SAR model. For any fixed $\boldsymbol{W}$, and for any $k \geq m_1$, there exists at least one k-dimensional subspace* $\mathrm{col}(\boldsymbol{X})$ *such that $\alpha^* = 1$.*

Remark 3.7 With regards to the statement from Krämer (2005) reported above, Theorem 3.6 (i) establishes that the statement is correct when $m_1 = 1$; (ii) provides a generalization to the case $m_1 > 1$; (iii) provides a generalization to POI tests and to CAR models.

From the proof of Theorem 3.6 it can be deduced that, in fact, for any $k \geq m_1$ there exists an infinity of subspaces $\mathrm{col}(\boldsymbol{X})$, such that $\alpha^* = 1$, i.e., such that the limiting power of the selected test vanishes for any α. Such "particularly hostile" subspaces $\mathrm{col}(\boldsymbol{X})$ are described in Appendix D, where the conclusion is reached that in CAR and symmetric SAR models it becomes impossible, as $\rho \to \lambda_{\max}^{-1}$, to detect large positive spatial autocorrelation when the regressors are the sum of a strongly positively autocorrelated component and a strongly negatively autocorrelated component.

Having established the existence of subspaces $\mathrm{col}(\boldsymbol{X})$ such that $\alpha^* = 1$, the question arises of how likely it is to run into them. One might suspect that the set of such $\mathrm{col}(\boldsymbol{X})$'s has measure zero (with respect to a suitable measure; see the next paragraph). We now show that such a suspicion is incorrect, because there is always, for any $\boldsymbol{W}$, any α, and any $n - k$, a nonzero probability that the limiting power of a POI or LBI test vanishes.

Let $G_{k,n}$ denote the set, usually called a Grassmann manifold, of all k-dimensional subspaces of $\mathbb{R}^n$, and let, for $0 < \alpha < 1$, $H_k(\alpha) \subseteq G_{k,n}$ be the set of k-dimensional $\mathrm{col}(\boldsymbol{X})$ such that the limiting power of a POI or LBI critical region of size less than α vanishes (for some CAR or symmetric SAR model). Then, as $\mathrm{col}(\boldsymbol{X})$ ranges over $G_{k,n}$ according to some probability distribution with respect to the invariant measure on $G_{k,n}$ (see James, 1954), we can define $z_\alpha = \Pr(\mathrm{col}(\boldsymbol{X}) \in H_k(\alpha))$. Such a probability can be interpreted as the *probability of zero limiting power* of a size-α POI or LBI test, in a CAR or symmetric SAR model.[9]

Theorem 3.8 *Consider, in the context of a CAR or symmetric SAR model, testing $\rho = 0$ against $\rho > 0$ by means of a POI or LBI test. If* $\mathrm{col}(\boldsymbol{X})$ *has density that is almost everywhere positive on $G_{k,n}$, with $k \geq m_1$, then $z_\alpha > 0$, for any $\boldsymbol{W}$ and regardless of how large α or $n - k$ is.*

Of course, in some circumstances z_α can be very small (e.g., for most combinations of distributions of $\mathrm{col}(\boldsymbol{X})$ and matrices $\boldsymbol{W}$, z_α is small when $n-k$ or α are large). The important message of Theorem 3.8 is that, under the stated conditions, z_α *is never zero*, or, put differently, in any practical case there is always a positive probability that the limiting power disappears.

3.3 Zero-Mean Models

We now specialize some of the above results to zero-mean CAR and SAR models (the extension to models with known and constant mean is trivial). The explicit consideration of the case $\boldsymbol{X} = \mathbf{O}$ serves two purposes. Firstly, it clarifies—by direct comparison with the regression case—the role played by the regressors in determining power. Secondly, it allows to focus on the effect of $\boldsymbol{W}$ on power.

Let us start from the following corollary of Theorem 3.3.

[9] Note that $H_k(\alpha_1) \subseteq H_k(\alpha_2)$ for any $\alpha_1 \geq \alpha_2$, and hence z_α is non-decreasing in α.

Corollary 3.9 *In zero-mean CAR and SAR models, the limiting power of an invariant critical region* $\Phi_{\boldsymbol{y}}$ *for testing* $\rho = 0$ *against* $\rho > 0$ *is* 1 *if* $\boldsymbol{f}_{\max} \in \Phi_{\boldsymbol{y}}$, 0 *otherwise.*

It is instructive to interpret Corollary 3.9 in terms of the Moran statistic $\boldsymbol{y}'\boldsymbol{W}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{y}$ associated to the zero-mean vector $\boldsymbol{y}$ (Moran, 1950). Observe that, when $\boldsymbol{W}$ is symmetric, $\boldsymbol{y}'\boldsymbol{W}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{y}$ achieves a maximum at $\boldsymbol{y} = \boldsymbol{f}_{\max}$. Thus, Corollary 3.9 says that, when $\boldsymbol{y}$ follows a zero-mean CAR or symmetric SAR model, an important requirement for an invariant critical region is that it contains the points in the sample space that lead to an high value of $\boldsymbol{y}'\boldsymbol{W}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{y}$. Rather surprisingly, the situation is not so intuitive in the case of asymmetric SAR models, because when $\boldsymbol{W}$ is nonsymmetric $\boldsymbol{y}'\boldsymbol{W}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{y}$ is not, in general, maximized by $\boldsymbol{f}_{\max}$. Thus, for the case of asymmetric SAR models, Corollary 3.9 implies that an invariant critical region may have vanishing limiting power even if it contains the values of $\boldsymbol{y}$ that maximize the Moran statistic.

The case of POI and LBI tests is emblematic of the role played by the symmetry of $\boldsymbol{W}$.

Proposition 3.10 *When* $\boldsymbol{X} = \mathbf{O}$ *and for a POI or LBI test,* $\alpha^* = 0$ *if and only if* $\boldsymbol{f}_{\max}$ *is an eigenvector of* $\boldsymbol{W}'$.

The condition in Proposition 3.10 is always satisfied when $\boldsymbol{W}$ is symmetric, i.e., for CAR and symmetric SAR models, but is generally not satisfied in the case of asymmetric SAR models (see Appendix E, where, in particular, it is shown that when $\boldsymbol{W}$ is row-stochastic the condition is satisfied if and only if $\boldsymbol{W}$ is doubly stochastic). Thus, Proposition 3.10 says that in a zero-mean CAR or symmetric SAR model the limiting power of POI and LBI tests is 1 for any α, whereas, typically, in a zero-mean asymmetric SAR model there are values of α (all values $\alpha \leq \alpha^* > 0$) such that the limiting power vanishes. The various possibilities for the limiting power of POI and LBI tests are summarized in Table 1. A simple example follows.

`Table 1 about here`

Example 3.11 A random variable is observed at n units placed along a line and, in the context of a zero-mean SAR process, it is to be tested whether $\rho = 0$ or $\rho > 0$. Suppose that it is believed that there is only first-order interaction, and that the interaction amongst first-order neighbors is stronger in one direction than in the other. Accordingly, $\boldsymbol{W}$ is chosen so that $W_{i,j}$, for $i, j = 1, ..., n$, is equal to some fixed positive scalar $w \neq 1$ if $i - j = 1$, to 1 if $j - i = 1$, and to 0 otherwise. In Figure 1, we plot the power function of the Moran test (i.e., the LBI test) for $\rho = 0$ against $\rho > 0$, and the envelope $\pi_\rho(\rho)$ for $n = 6$, $w = 10$ and $\alpha = 0.01$. The power has been computed numerically, via the Imhof method (Imhof, 1961), and is plotted against $\rho\lambda_{\max}$, which ranges between 0 and 1.

`Figure 1 about here`

Although it is based on a model with an artificial $\boldsymbol{W}$, Figure 1 illustrates the theoretically important point that in an asymmetric SAR model the performance of the Moran test may be extremely disappointing, even when the model is not contaminated by regressors.

Note that the possibility of a limiting zero power of the Moran test has also implications for the interpretation of the Moran statistic as an index of autocorrelation. For instance, in the case of Example 3.11, Figure 1 shows that there exist values $0 < k < \lambda_{\max}^{-1}$ such that $\Pr(\boldsymbol{y}'\boldsymbol{W}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{y} > k)$ is not increasing in ρ over the interval $(0, \lambda_{\max}^{-1})$, although, as it is easily checked, all correlations implied by the model are increasing in ρ over the interval $(0, \lambda_{\max}^{-1})$ (Proposition 4.2 below states that, on the contrary, when $\boldsymbol{W}$ is symmetric the power function of the Moran test is monotonic). In simple terms, in an asymmetric SAR model the probability of the Moran statistic being large may be very small, even if all correlations are high. It is clear that in these circumstances the interpretation of the Moran statistic as an index of autocorrelation cannot be justified.

Having argued that, generally, in zero-mean asymmetric SAR models $\alpha^* > 0$, a practically important question is which (nonsymmetric) matrices $\boldsymbol{W}$ are associated to large values of α^*. Let us return for a moment to our example of a SAR model defined on a line.

Example 3.12 For the case of Example 3.11, the Imhof method (or some other numerical approximation to the null distribution of the Moran statistic) can be used to verify that α^* is decreasing in n and increasing in $|w-1|$. For the particular case of Figure 1, α^* is about 0.056, i.e., any Moran critical region of size less than 0.056 has vanishing limiting power. To give another example, if $n = 30$ and $w = 50$, then α^* is about 0.063. Interestingly, if one closes the line (i.e., if one sets $W_{1,n} = w$ and $W_{n,1} = 1$), then $\boldsymbol{W}$ becomes a scalar multiple of a doubly stochastic matrix, and consequently $\alpha^* = 0$ by Corollary E.1.

Numerical investigations not reported here show that the message delivered by Example 3.12 is very general. Namely, large values of α^* are typically associated to small nonsymmetric $\boldsymbol{W}$, or, for a fixed n, to matrices $\boldsymbol{W}$ such that $W_{i,j}/W_{j,i}$ is large for at least one pair (i,j). This suggests that the asymmetry introduced by using row-standardized $\boldsymbol{W}$'s (see Appendix A) does not yield large values of α^* in zero-mean SAR models, because for such matrices $W_{i,j}/W_{j,i} \leq u(\boldsymbol{A})$, $i,j = 1,...,n$, where $u(\boldsymbol{A})$ denotes the ratio of the largest to the smallest row-sum of $\boldsymbol{A}$.[10] Note that the largest possible value of $u(\boldsymbol{A})$ over all $n \times n$ matrices $\boldsymbol{A}$ is $n-1$, obtained for the adjacency matrix of a star graph (i.e., a graph with one vertex having $n-1$ neighbors, and all other vertices having 1 neighbor). Even in the case of a star graph, the value of α^* associated to the corresponding row-standardized $\boldsymbol{W}$ is very small, and decreasing in n. For instance, for the Moran test, when $\boldsymbol{W}$ is the row-standardized version of the adjacency matrix of a star graph, $\alpha^* > 0.01$ only when $n < 6$ (i.e., the limiting power of the Moran test is 1 as long as $n \geq 6$ and $\alpha > 0.01$). We can thus conclude that, although in SAR models asymmetry of $\boldsymbol{W}$ may cause the limiting power of POI and LBI tests to disappear even when $\boldsymbol{X} = \mathbf{O}$, in practice this typically occurs only for very small values of α or n when the asymmetry of $\boldsymbol{W}$ is due to row-standardization of a symmetric matrix. As we have seen in Section 3.2, when $\boldsymbol{X} \neq \mathbf{O}$ the situation is very different and, for any $\boldsymbol{W}$, the limiting power of POI and LBI may vanish even for large α. Examples with $\boldsymbol{X} \neq \mathbf{O}$ are given next.

3.4 Numerical Examples

In this subsection we report results from a small Monte Carlo experiment aimed at illustrating how $\boldsymbol{X}$ and $\boldsymbol{W}$ affect the exact power of tests for residual spatial autocorrelation. More specifically, the objective is to show how sensitive the power of tests of spatial autocorrelation can be to $\boldsymbol{X}$, for a fixed $\boldsymbol{W}$ and when ρ is large but not necessarily very close to $\lambda_{\max}^{-1}$. For simplicity, we restrict attention to the power, to be denoted by $\pi_{LBI}(\rho)$, of the Cliff-Ord test in the context of a SAR model. Related numerical investigations are contained in Krämer (2005).

[10] It is interesting to note that the effect of the asymmetry of a row-standardized weight matrix $\boldsymbol{D}^{-1}\boldsymbol{A}$ (or any other non-symmetric matrix that is similar to a symmetric matrix) can always be eliminated by suitably selecting $\boldsymbol{V}$ in (3). In fact, model (3) with $\boldsymbol{W} = \boldsymbol{D}^{-1}\boldsymbol{A}$ and $\boldsymbol{V} = \boldsymbol{D}^{-1}$ is reduced, upon normalization to $\boldsymbol{\Sigma}(0) = \boldsymbol{I}$, to a SAR model with the symmetric weight matrix $\boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2}$.

We consider 10^6 replications of the $n \times 2$ matrix $\boldsymbol{X} = (\boldsymbol{\iota} : \boldsymbol{x})$, where $\boldsymbol{\iota}$ is the n-dimensional vector of all ones and $\boldsymbol{x} \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{I})$.[11] The weights matrices are derived from the maps of the $n = 17$ counties of Nevada and the $n = 23$ counties of Wyoming. We consider both a binary $\boldsymbol{W}$, specified according to the queen criterion (i.e., $W_{i,j} = 1$ if two distinct counties i and j share a common boundary or a common point, $W_{i,j} = 0$ otherwise), and its row-standardized version. The average number of neighbors of a county is 4.35 in Nevada, 4.52 in Wyoming, whereas the sparseness of $\boldsymbol{W}$ (as measured by the percentage of zero entries) is 74.40 for Nevada and 80.34 for Wyoming. We shall see that, despite their similarities, these two spatial configurations are very different from the point of view of testing for autocorrelation.

Firstly, in order to show how sensitive $\pi_{LBI}(\rho)$ is to $\boldsymbol{X}$, in Table 2 we display the percentage frequency distribution of $\pi_{LBI}(\rho)$, with $\boldsymbol{W}$ as described above. The size α is set to 0.05, and power is computed by the Imhof method. We report values for $\rho = 0.9\lambda_{\max}^{-1}$ and $\rho = 0.95\lambda_{\max}^{-1}$. To give an indication of how close such points are to $\lambda_{\max}^{-1}$, the third column of Table 2 gives the average correlation between pairs of neighboring counties (there are 37 such pairs in Nevada and 54 in Wyoming; averages over non-neighbors, not reported, are much lower). Note that any correlation implied by a SAR model tends to 1 as $\rho \to \lambda_{\max}^{-1}$.[12] By Theorem 3.3, in our experiment $\lim \pi_{LBI}(\rho)$ (as $\rho \to \lambda_{\max}^{-1}$) is either 0 or 1 when $\boldsymbol{W}$ is binary (as, in that case, $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$ almost surely), whereas it is in $(0, 1)$ when $\boldsymbol{W}$ is row-standardized (as, in that case, $\boldsymbol{f}_{\max} \in \mathrm{col}(\boldsymbol{X})$; cf. Remark F.2). It appears from Table 2 that in the case of Nevada $\pi_{LBI}(\rho)$ depends to a very large extent on $\boldsymbol{X}$, even at points that are not in a very small neighborhood of $\lambda_{\max}^{-1}$. The dependence is less pronounced in the case of Wyoming.

`Table 2 about here`

Next, we consider the zero limiting power problem more closely, which requires restricting attention to binary weights matrices (so that $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$ almost surely). In Table 3 we display $z_{0.05}$ (see Section 3.2), obtained as the observed frequency of times that (12) (with c_α computed by the Imhof method) is positive in our experiment. Note that $z_{0.05}$ is very large in the case of Nevada, whereas it is very small in the case of Wyoming. The table also displays the average shortcoming (i.e., $\pi_\rho(\rho) - \pi_{LBI}(\rho)$) of the Cliff-Ord test at $\rho = 0.9\lambda_{\max}^{-1}$ and $\rho = 0.95\lambda_{\max}^{-1}$, when $\lim \pi_{LBI}(\rho) = 0$ and when $\lim \pi_{LBI}(\rho) = 1$. It appears that the impact of the zero limiting power problem is not localized only in a very small neighborhood of $\lambda_{\max}^{-1}$, because, on average, an $\boldsymbol{X}$ such that $\lim \pi_{LBI}(\rho) = 0$ causes shortcomings at $\rho = 0.9\lambda_{\max}^{-1}$ and $\rho = 0.95\lambda_{\max}^{-1}$ that are significantly larger than the corresponding shortcomings associated to an $\boldsymbol{X}$ such that $\lim \pi_{LBI}(\rho) = 1$.

`Table 3 about here`

We now comment on how the specific cases just analyzed may be representative of more general situations. The most important concept here is that z_α is generally very sensitive to $\boldsymbol{W}$, n, k, the choice of a test, α, and the distribution of $\boldsymbol{X}$. For most matrices $\boldsymbol{W}$ likely to be used in applications and for most distributions of $\boldsymbol{X}$, z_α is generally small when $n - k$ is large. This suggests that, from a practical point of view, the zero limiting power problem is mainly a small sample problem. It should

[11] Observe that, because of its invariance property, the power of the Cliff-Ord test depends on $\boldsymbol{X}$ only through $\mathrm{col}(\boldsymbol{X})$. Thus, it would be natural to draw $\boldsymbol{X}$ from $\mathrm{N}(\boldsymbol{0}, \boldsymbol{I}_n \otimes \boldsymbol{I}_k)$, as this would imply that $\mathrm{col}(\boldsymbol{X})$ is uniformly distributed on the Grassmann manifold $G_{k,n}$ (see James, 1954). In our simulations, we have modified such a distribution to take into account the fact that, in practice, an intercept is always included in the regression.

[12] This is because a SAR model tends to be concentrated on a 1-dimensional subspace of the sample space (see Appendix C).

however be noticed that z_α is always positive, regardless of n, under the condition in Theorem 3.8, and that it is possible to construct matrices $\boldsymbol{W}$ (e.g., the adjacency matrix of a star graph or a very dense matrix) such that, for some distributions of $\boldsymbol{X}$, z_α is large even when $n-k$ is large. Another interesting point worth emphasizing is that, in general, z_α is significantly larger when the regression includes an intercept. This is because, due to the nonnegativity of $\boldsymbol{W}$, $\boldsymbol{\iota}$ usually (and especially if the row sums of $\boldsymbol{W}$ are all of similar magnitude) yields a large value of the Moran statistic, and therefore its presence tends to put more probability mass on subspaces $\mathrm{col}(\boldsymbol{X})$ close to the ones defined by Proposition D.1. Finally, observe that when $\boldsymbol{W}$ is defined on a regular grid, one can study how z_α depends on n explicitly (cf. Table 1 of Krämer, 2005).

To summarize, the main conclusion of our numerical study is that, in some cases, the probability that the limiting power of the Cliff-Ord test vanishes may well be non-negligible. This obviously induces a large dependence of the power of the Cliff-Ord test on $\boldsymbol{X}$ when $\rho \to \lambda_{\max}^{-1}$, but the numerical results indicate that both the power and the shortcoming may still depend to a large extent on $\boldsymbol{X}$ for values of ρ in a rather large neighborhood of $\lambda_{\max}^{-1}$. As we have already mentioned above, this is cause of concern, because such values may induce a large inefficiency of the ordinary least squares estimator of β.

4 Unbiasedness and Monotonicity

In this section we turn our attention to global power properties of invariant critical regions for the testing problem (6). In particular, we discuss some conditions on model $\mathrm{N}(\boldsymbol{X\beta}, \sigma^2\boldsymbol{\Sigma}(\rho))$ that are sufficient for POI and LBI tests to be unbiased (for a general $\boldsymbol{\Sigma}(\rho)$) and to have power functions monotonic in ρ (for the $\boldsymbol{\Sigma}(\rho)$ implied by CAR or SAR models). The conditions are not necessary, but, as we shall see, (i) are important to understand the structure of the testing problem under analysis; (ii) in the case of spatial autoregressive models, admit a simple interpretation.

One case in which POI and LBI tests for $\rho = 0$ against $\rho > 0$ in $\mathrm{N}(\boldsymbol{X\beta}, \sigma^2\boldsymbol{\Sigma}(\rho))$ are certainly unbiased is when $\boldsymbol{X}$ and $\boldsymbol{\Sigma}(\rho)$ are such that a UMPI test exists (see Remark G.3). This is however a very restrictive condition, in that it requires the critical region defined by (8) to be independent of $\bar{\rho}$. We now formulate two conditions that, when taken together, guarantee unbiasedness of POI and LBI tests, even in cases when a UMPI test does not exist. Following Horn and Johnson (1985), a commuting family of matrices is a finite or infinite set of matrices that are pairwise commutative under standard multiplication.

Condition A *The matrices $\boldsymbol{\Sigma}(\rho)$, for $\rho > 0$, form a commuting family.*

Condition B *For a fixed $\widetilde{\rho} > 0$, $\mathrm{col}(\boldsymbol{X})$ is spanned by k linearly independent eigenvectors of $\boldsymbol{\Sigma}(\widetilde{\rho})$.*

It is easily checked that Condition A is satisfied by CAR and symmetric SAR models, but not by asymmetric SAR models (except for very special cases). A well-known characterization of a commuting family of symmetric matrices is that all its members share the same eigenvectors. Thus, when Condition A holds, Condition B does not depend on $\widetilde{\rho}$. Condition B, in any of its many equivalent formulations, has been often used in the theoretical analysis of regression models with non-spherical errors, since Anderson (1948). Although it is unlikely to be met in practice, in some circumstances Condition B may be expected to be satisfied approximately; see the end of this section for CAR and symmetric SAR models, and Durbin (1970) for the case of serial correlation. There is evidence in the literature that the power properties of tests for $\rho = 0$ when Condition B holds exactly are similar to those when Condition B holds approximately (e.g., Tillman, 1975, p. 971).

Letting $\mathrm{col}^{\perp}(\boldsymbol{X})$ denote the orthogonal complement of $\mathrm{col}(\boldsymbol{X})$, we obtain:[13]

Proposition 4.1 *Assume that Conditions A and B hold. Then, any POI or LBI test for $\rho = 0$ against $\rho > 0$ in model $\mathrm{N}(\boldsymbol{X\beta}, \sigma^2\boldsymbol{\Sigma}(\rho))$ is unbiased. The unbiasedness is strict except when $\mathrm{col}^{\perp}(\boldsymbol{X})$ is a subset of an eigenspace of $\boldsymbol{\Sigma}(\rho)$, in which case the power is α for any $\rho > 0$.*

It is important to note that, while guaranteeing unbiasedness of the tests considered in Proposition 4.1, Conditions A and B are not sufficient for the monotonicity of the power functions of those tests (not even when $\boldsymbol{X} = \mathbf{O}$). This is because, given a $\boldsymbol{\Sigma}(\rho)$ satisfying Condition A, a reparametrization $\rho \to f(\rho)$ may destroy the monotonicity of the power function without causing Condition A to fail. In fact, monotonicity of the power function in ρ is a much stronger property than unbiasedness, and may or may not be desirable depending on the specification of $\boldsymbol{\Sigma}(\rho)$. In general, it is desirable whenever ρ is interpreted as an autocorrelation parameter. This is the case for CAR and SAR models.

Proposition 4.2 *Consider testing $\rho = 0$ against $\rho > 0$ in a CAR or symmetric SAR models satisfying Condition B for one value (and hence all values) $\widetilde{\rho} > 0$. The power function of any POI and LBI test for $\rho = 0$ against $\rho > 0$ is non-decreasing. It is strictly increasing except when $\mathrm{col}^{\perp}(\boldsymbol{X})$ is a subset of an eigenspace of $\boldsymbol{W}$, in which case the power is α for any $\rho > 0$.*

Proposition 4.2 implies that, in CAR and symmetric SAR models satisfying Condition B, LBI and POI test statistics can be regarded as indexes of autocorrelation, because the probability of them being greater than some constant (i.e. the power of the associated tests) is non-decreasing (as all correlations between pairs of variables y_i and y_j) in ρ, over $(0, \lambda_{\max}^{-1})$. On the other hand, when Condition B is not satisfied, the power of the tests can be non-monotonic, which makes it difficult to interpret the underlying test statistics as indexes of autocorrelation (cf. Section 3.3). In Appendix G we report an additional result, Proposition G.8, that links Proposition 4.2 to the analysis of Section 3.

Next, we attempt to understand in which cases Condition B can be expected to hold at least approximately, in the context of CAR and SAR models (for examples of cases in which Condition B is satisfied exactly see Remark G.6). We consider, for simplicity, the case when there is only one regressor, denoted by $\boldsymbol{x} = (x_1, ..., x_n)'$. Call two units i and j neighbors if $W_{i,j} > 0$, and let $\bar{x}_i := \sum_{j \neq i} W_{i,j} x_j$ be the weighted average of the values of $\boldsymbol{x}$ observed at units that are neighbors of i. Then, the ratio $x_i/\bar{x}_i$ may be regarded as a measure of "similarity" between unit i and its neighbors (as far as $\boldsymbol{x}$ is concerned). Further, observe that in CAR and symmetric SAR models Condition B is met if and only if the ratio $x_i/\bar{x}_i$ does not depend on i, because in such models the eigenvectors of $\boldsymbol{\Sigma}(\rho)$ are the same as those of $\boldsymbol{W}$ and, by the definition of an eigenvector of $\boldsymbol{W}$, the ratio $x_i/\bar{x}_i$ must, for any i, be equal to the corresponding eigenvalue. We may therefore conclude that Condition B is approximately met, and hence the power of POI and LBI invariant tests has desirable properties, when $\boldsymbol{x}$ is such that the degree of similarity between i and its neighbors does not vary substantially with i.[14] On the contrary, Condition B is far from being satisfied when different clusters of neighbors

[13] In Proposition 4.1, as in Propositions 4.2 and G.8, "any" means for any $\bar{\rho}$ and any α.

[14] Note that the fact that the degree of similarity between i and its neighbors does not change substantially with i does not imply that $\boldsymbol{x}$ is highly autocorrelated. Suppose, for simplicity, that $\mathrm{E}(\boldsymbol{x}) = \mathbf{0}$, so that the Moran statistic associated to it is $\boldsymbol{x}'\boldsymbol{W}\boldsymbol{x}/\boldsymbol{x}'\boldsymbol{x}$. Then, when Condition B is satisfied, i.e. $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{W}$, $\boldsymbol{x}'\boldsymbol{W}\boldsymbol{x}/\boldsymbol{x}'\boldsymbol{x}$ equals

have very different degrees of similarity.

5 Discussion

This paper has investigated a number of exact properties of invariant tests for autocorrelation in the context of a linear regression model with errors following a (first-order) conditional or simultaneous spatial autoregressive process. The main message of our analysis is that the power properties of tests for residual spatial autocorrelation may depend to a great extent on the regressors, especially when the number of degrees of freedom is small and the autocorrelation is large. Intuitively, this is largely due to the fact that CAR and SAR models tend, as the autocorrelation increases, to a family of (improper) distributions supported on a 1-dimensional subspace of the sample space. In particular, we have characterized the circumstances when the regressors are such that the intersection between the above subspace and a critical region has 1-dimensional Lebesgue measure zero; in this case the power of the critical region vanishes as the autocorrelation increases.

In order to derive our exact results, we have worked under some potentially restrictive assumptions. More specifically, for the case of SAR models, we have not allowed for spatial lags $\boldsymbol{W}\boldsymbol{y}$ of the dependent variable amongst the regressors (see, e.g., Ord, 1975, and Lee, 2002), we have assumed homoskedasticity of the innovations (i.e., $\boldsymbol{V}$ in (3) is fixed), we have committed ourselves to elliptically symmetric distributions.[15] In future work, it would certainly be of interest to study exact properties of tests for spatial autocorrelation when such assumptions are relaxed. The main obstacle to this type of extensions is that invariance would no longer be sufficient to eliminate all nuisance parameters, thus originating the need to resort to different techniques, or to (asymptotic) approximations.

Two more straightforward extensions of our analysis are as follows. First, while in this paper we have mostly focused on the power as $\rho \to \lambda_{\max}^{-1}$, the techniques we have used should also prove useful to study local power, by studying the right derivative of the power function at $\rho = 0$. Second, the results in the paper relative to POI and LBI tests can be extended to allow for misspecification of $\boldsymbol{W}$ (cf. Kelejian and Prucha, 2001, p. 225). In the paper, we have assumed, for simplicity, that POI and LBI test statistics are based on a weights matrix that is the same as the one that has generated the data, but it is clear that misspecification of $\boldsymbol{W}$ may be a practically important issue.[16]

Appendix A The Weights Matrices

The weights matrices $\boldsymbol{W}$ used in CAR and SAR models are assumed to satisfy the following three conditions: (i) $W_{i,i} = 0$, for $i = 1, ..., n$; (ii) $W_{i,j} \geq 0$, for $i, j = 1, ..., n$; (iii) $\boldsymbol{W}$ is an irreducible matrix (e.g., Gantmacher, 1974, Ch. 13). Condition (i) is required for the validity of CAR models (e.g., Besag, 1974), and is assumed for SAR models merely for convenience. Condition (ii) is not required by the definition of the models, but is virtually always satisfied in empirical applications. Condition (iii) is a natural assumption in a spatial context. An $n \times n$ matrix is said to be reducible if its index set $\{1, ..., n\}$ can be partitioned into two disjoint and non-empty sets U_1 and U_2 such that $A_{i,j} = 0$, for $i \in U_1$, $j \in U_2$; it is said to be *irreducible* if it is not reducible. In graph theoretic terms,

an eigenvalue of $\boldsymbol{W}$.

[15] Note that none of these restrictions applies to CAR models, in the sense that CAR models do not allow for such extensions.

[16] Extensions to cover this type of misspecification can be based on the fact that the matrix $\boldsymbol{B}$ in Lemma F.1 does not need to be a function of the weights matrix appearing in the data generating process.

irreducibility of a weights matrix amounts to requiring that the graph with adjacency matrix $\boldsymbol{W}$ (that is, the graph with the n observational units as vertices and an edge from i to j if and only $W_{i,j} \neq 0$) is *strongly connected*, i.e., has a path between any two distinct vertices (e.g., Cvetković *et al.*, 1980, p. 18). Observe that (ii) implies that (non-circular) AR(1) models are not in our class of SAR processes, because the matrix $\boldsymbol{W}$ necessary to write the covariance matrix of an AR(1) process as in equation (3) would be triangular and hence reducible. (We also mention that extensions of condition (iii) to block diagonal weights matrices with irreducible blocks, as for instance in Case, 1991, are possible, but not pursued here for simplicity.)

In addition to the three assumptions above, recall from Section 2.1 that $\boldsymbol{W}$ is assumed to be symmetric in CAR models, but can be nonsymmetric in SAR models. The most popular nonsymmetric weights matrices in applications are those obtained by row-standardizing a preliminary matrix (e.g., Anselin, 1988). Formally, by a *row-standardized* $\boldsymbol{W}$ we mean one that can be written as $\boldsymbol{W} = \boldsymbol{D}^{-1}\boldsymbol{A}$, where $\boldsymbol{A}$ is a symmetric $(0,1)$ matrix (i.e., a matrix containing only zeroes and ones) and $\boldsymbol{D}$ is the diagonal matrix with $D_{i,i} = \sum_{j=1}^{n} A_{i,j}$, $i = 1, ..., n$.

Assumptions (ii) and (iii) have the advantage of making the *Perron-Frobenius theorem* for nonnegative irreducible matrices (e.g., Gantmacher, 1974) available to derive information about the spectral properties of weights matrices. Let $\lambda_{\max}$ denote the largest eigenvalue in modulus of $\boldsymbol{W}$. Important implications of the Perron-Frobenius theorem are: (a) $\lambda_{\max}$ is (real) positive; (b) $\lambda_{\max}$ has algebraic and geometric multiplicity 1; (c) there exists an entrywise positive eigenvector associated to $\lambda_{\max}$. In the paper, the unique entrywise positive and normalized (according to the Euclidean norm) eigenvector of $\boldsymbol{W}$ pertaining to $\lambda_{\max}$ is denoted by $\boldsymbol{f}_{\max}$.

Observe that both covariance matrices of CAR and SAR models are not defined at $\rho = \lambda_{\max}^{-1}$. In the paper we assume $\rho < \lambda_{\max}^{-1}$. Such a restriction is necessary for positive definiteness of the covariance matrix of a CAR model. For a SAR model, it is not necessary, but has the advantage of guaranteeing connectedness of $\mathbb{R}^{+} \cap \Psi$ and of avoiding undesirable behavior of the covariances implied by the model when $\rho > \lambda_{\max}^{-1}$. When $0 < \rho < \lambda_{\max}^{-1}$, it is easily established (e.g., using eq. (55) of Gantmacher, 1974) that conditions (ii) and (iii) imply that, in both CAR and SAR, models $\mathrm{cov}(y_i, y_j) > 0$ for any $i, j = 1, ..., n$ (similarly, it can be shown that when $\rho < 0$ the covariances may be positive or negative, but not all of them are positive in a left neighborhood of $\lambda_{\max}^{-1}$). Thus, for CAR and SAR models, the alternative hypothesis $\rho > 0$ represents positive spatial autocorrelation.

Appendix B Definition of Invariant Tests

Let $F_{\boldsymbol{X}}$ the group of transformations $\boldsymbol{y} \to a\boldsymbol{y} + \boldsymbol{X}\boldsymbol{b}$, with $a \in \mathbb{R}\backslash\{0\}$ and $\boldsymbol{b} \in \mathbb{R}^{k}$, and let $F_{\boldsymbol{X}}^{+}$ be the smaller group obtained when $a > 0$. In this paper, invariant tests are defined with respect to $F_{\boldsymbol{X}}$, as for instance in Berenblut and Webb (1973). More commonly, invariance of tests for autocorrelation in linear regression is defined with respect to $F_{\boldsymbol{X}}^{+}$ (see, e.g., King, 1988). Under $F_{\boldsymbol{X}}^{+}$, a maximal invariant is $\boldsymbol{C}\boldsymbol{y}/\left\|\boldsymbol{C}\boldsymbol{y}\right\|$, and invariant critical regions are defined on the unit $(n-k)$-sphere (rather than on an hemisphere). The class of critical regions that are invariant under $F_{\boldsymbol{X}}$ is equivalent to the class of critical regions that are invariant under $F_{\boldsymbol{X}}^{+}$ and are centrally symmetric (i.e., they contain a vector $\boldsymbol{t}$ if and only if they contain $-\boldsymbol{t}$). The distinction between the two classes of tests, although needed for the results in Section 3, is not substantive, because tests that are invariant under $F_{\boldsymbol{X}}^{+}$ but not under $F_{\boldsymbol{X}}$ are never used in practice. (Although a formal proof is not attempted here, it should be possible to prove that any test that is invariant under $F_{\boldsymbol{X}}^{+}$ but not under $F_{\boldsymbol{X}}$ is inadmissible, i.e., dominated uniformly over ρ by a test that is invariant under $F_{\boldsymbol{X}}$.)

Finally, it is worth clarifying the relationship between invariant and similar tests for testing problem (6). In Section 2.2, it was noted that invariant tests are similar. In fact a much stronger property holds: the class of tests that are invariant with respect to $F_{\boldsymbol{X}}^{+}$ coincides with the class of similar tests for $\rho = 0$ (see Hillier, 1987), and the class of tests that are invariant with respect to $F_{\boldsymbol{X}}$ coincides with the class of centrally symmetric similar tests for $\rho = 0$.

Appendix C A Geometrical Interpretation of Lemma 3.2

In the context of Lemma 3.2, if $a \in \Psi$, then $\lambda_{n-k}(\boldsymbol{\Omega}) < \infty$ and thus, as one would expect, the limit of the power function is in $(0, 1)$. A more interesting case is when a is an accumulation point of Ψ that is not in Ψ; then, the limit of the power function can be anywhere in $[0, 1]$. Let $\varphi = n - \lim_{\rho \to a} \{\mathrm{rk}(\boldsymbol{\Sigma}^{-1}(\rho))\}$. Lemma 3.2 asserts that, for the power to vanish as $\rho \to a$, $\lambda_{n-k}(\boldsymbol{\Omega})$ must be infinite, which requires $\varphi > 0$. Now, when $\varphi > 0$, model $\mathrm{N}(\boldsymbol{X\beta}, \sigma^2 \boldsymbol{\Sigma}(\rho))$ tends, as $\rho \to a$, to a family of (improper) distributions supported on a φ-dimensional subspace, say Λ_φ, of $\mathbb{R}^n$. Geometrically, this is apparent from the fact that the contours of $\mathrm{N}(\boldsymbol{X\beta}, \sigma^2 \boldsymbol{\Sigma}(\rho))$ are the ellipsoids $(\boldsymbol{y} - \boldsymbol{X\beta})' \boldsymbol{\Sigma}^{-1}(\rho)(\boldsymbol{y} - \boldsymbol{X\beta}) = c$ in $\mathbb{R}^n$, for any $c > 0$, which also shows that, for any fixed $\boldsymbol{\beta}$, Λ_φ is the translation of $\lim_{\rho \to a} \{E_n(\boldsymbol{\Sigma}(\rho))\}$ by $\boldsymbol{X\beta}$ (recall from the Introduction that normality is assumed only for convenience, and can be replaced by elliptical symmetry). Thus, as $\rho \to a$, the power of a certain test depends on the position of the critical region in $\mathbb{R}^n$ relative to Λ_φ. In particular, the power of a test vanishes whenever the intersection between Λ_φ and the critical region has 1-dimensional Lebesgue measure zero. Note that this statement holds for any test, whether it is invariant or not; if the test is invariant, then the position of the critical region relative to Λ_φ does not depend on $\boldsymbol{\beta}$, and the conditions in Lemma 3.2 can be exploited. Our argument makes it clear the the vanishing of the limiting power cannot be attributed to the specific form of a particular test statistic (as argued, for instance, in Krämer 2005, p. 490), but is a consequence of a property of the statistical model.

The condition $\varphi > 0$ required for the power to vanish as $\rho \to a$ emphasizes an important difference between spatial and time-series autoregressions (recall from Appendix A that, due to the assumption of irreducibility of $\boldsymbol{W}$, the latter are not included in the former). On the one hand, when $a = \lambda_{\max}^{-1}$ in CAR and SAR models, $\varphi = 1$ for any $\boldsymbol{W}$, and hence the vanishing of the power as $\rho \to a$ can always occur. On the other hand, when $a = 1$ in a (finite) AR(1) model, in order for $\varphi = 0$ and hence for the power as $\rho \to a$ to vanish, it is necessary that the initial condition is specified in such a way that the model is covariance stationary.

Remark C.1 In the case of an AR(1) model, the power of the Durbin-Watson and some related tests as $\rho \to 1$ has been investigated extensively; see Krämer (1985), Zeisel (1989) and Bartels (1992). Lemma 3.2 shows that the results in those papers can be easily extended to any invariant test for residual serial correlation.

Remark C.2 A result similar to Lemma 3.2 can be obtained when $\boldsymbol{\Sigma}(\rho)$, rather than $\boldsymbol{\Sigma}^{-1}(\rho)$, tends to a singular matrix as $\rho \to a$. In that case, as $\rho \to a$, the distributions $\mathrm{N}(\boldsymbol{X\beta}, \sigma^2 \boldsymbol{\Sigma}(\rho))$, for any $\boldsymbol{\beta}$, are supported on a subspace of $\mathbb{R}^n$ of dimension $\lim_{\rho \to a} \{\mathrm{rk}(\boldsymbol{\Sigma}(\rho))\}$. Examples include a spatial moving average model (e.g., Anselin, 1988), and a fractionally integrated white noise, with ρ being the differencing parameter and $a = 1/2$ (see Kleiber and Krämer, 2005).

Appendix D The Particularly Hostile $\mathrm{col}(\boldsymbol{X})$'s

In this appendix we provide an interpretation of the subspaces $\mathrm{col}(\boldsymbol{X})$ that, in the context of Theorem 3.6, are "particularly hostile" from the point of view of testing for spatial autocorrelation, in that they yield $\alpha^* = 1$. We limit ourselves to the case $k = m_1$, because, as is clear from Theorem 3.6, the case $k > m_1$ is trivial. The case $k = m_1 = 1$ is the simplest, and is completely characterized by the following result.

Proposition D.1 *Consider a CAR or symmetric SAR model containing one single regressor, equal to $\boldsymbol{f}_1(\boldsymbol{W}) + \omega \boldsymbol{f}_{\max}$, for some $\omega \in \mathbb{R}$, and suppose that $m_1 = 1$. Then, the limiting power a POI or LBI test for $\rho = 0$ against $\rho > 0$ vanishes for any α if $|\omega| \geq \omega^*$, where ω^* is a threshold that depends on the model and on the test. Namely, letting*

$$\omega_1 = \left[\frac{\lambda_{\max} - \lambda_2(\boldsymbol{W})}{\lambda_2(\boldsymbol{W}) - \lambda_1(\boldsymbol{W})}\right]^{\frac{1}{2}}, \ \omega_2 = \frac{1 - \bar{\rho}\lambda_1(\boldsymbol{W})}{1 - \bar{\rho}\lambda_{\max}}, \ \omega_3 = \frac{2 - \bar{\rho}(\lambda_{\max} + \lambda_2(\boldsymbol{W}))}{2 - \bar{\rho}(\lambda_2(\boldsymbol{W}) + \lambda_1(\boldsymbol{W}))},$$

ω^* is equal to $\omega_1\sqrt{\omega_2}$ for a POI test in a CAR model, $\omega_1\omega_2\omega_3$ for a POI test in a symmetric SAR model, ω_1 for a LBI test in both models.

Proof of Proposition D.1. The result can be proved following the proof of Theorem 3.6 with $k = m_1 = 1$. When $m_1 = 1$ and $\boldsymbol{X}$ is a vector proportional to $\boldsymbol{f}_1(\boldsymbol{W}) + \omega \boldsymbol{f}_{\max}$, $\mathrm{col}(\boldsymbol{X}) \in \Theta$ as long as $\omega \neq 0$. Then, for the LBI test, we need to establish which values of ω yield $\tilde{\lambda} \leq \lambda_2(\boldsymbol{W})$. Recalling that the vectors $\boldsymbol{f}_1(\boldsymbol{W})$ and $\boldsymbol{f}_{\max}$ are normalized, it is easily checked that $\boldsymbol{M_X f}_{\max} = [\boldsymbol{f}_{\max} - \omega \boldsymbol{f}_1(\boldsymbol{W})]/(1+\omega^2)$. Plugging such an expression in (15), and using the fact that $\boldsymbol{C f}_1(\boldsymbol{W}) = -\omega \boldsymbol{C f}_{\max}$ (since $\boldsymbol{CX} = \mathbf{O}$), we obtain $\tilde{\lambda} = [\lambda_{\max} + \omega^2 \lambda_1(\boldsymbol{W})]/(1+\omega^2)$. Hence, $\tilde{\lambda} \leq \lambda_2(\boldsymbol{W})$ requires $|\omega| \geq [\lambda_{\max} - \lambda_2(\boldsymbol{W})]^{1/2} [\lambda_2(\boldsymbol{W}) - \lambda_1(\boldsymbol{W})]^{-1/2}$, proving the part of the proposition relative to the LBI test. By straightforward extension, the limiting power of a POI test disappears for any α if $|\omega| \geq \{\lambda_n[\boldsymbol{\Sigma}(\bar{\rho})] - \lambda_2[\boldsymbol{\Sigma}(\bar{\rho})]\}^{1/2}\{\lambda_2[\boldsymbol{\Sigma}(\bar{\rho})] - \lambda_1[\boldsymbol{\Sigma}(\bar{\rho})]\}^{-1/2}$. The proof of the proposition is then completed on substituting $\lambda_i[\boldsymbol{\Sigma}(\bar{\rho})] = [1 - \bar{\rho}\lambda_i(\boldsymbol{W})]^{-r}$ in the last inequality, with $r = 1$ for a CAR model and $r = 2$ for a symmetric SAR model. ∎

Proposition D.1 defines the set of 1-dimensional $\mathrm{col}(\boldsymbol{X})$'s such that $\alpha^* = 1$. For a POI test, it is easily seen that, as $\bar{\rho}$ increases, this set becomes smaller and more concentrated in the direction of $\boldsymbol{f}_{\max}$. More generally, it can be deduced from the proof of Theorem 3.6 that, when $k = m_1 \geq 1$, the particularly hostile $\mathrm{col}(\boldsymbol{X})$'s are subspaces belonging to the span of $\boldsymbol{f}_1(\boldsymbol{W}), ..., \boldsymbol{f}_{m_1}(\boldsymbol{W}), \boldsymbol{f}_{\max}$. Consider the Moran statistic $\boldsymbol{x}'\boldsymbol{W}\boldsymbol{x}/\boldsymbol{x}'\boldsymbol{x}$ associated to a (zero-mean, for simplicity) vector $\boldsymbol{x} \in \mathbb{R}^n$ and a symmetric $\boldsymbol{W}$ (the standard version of the Moran statistic would include a normalizing factor and a correction for the sample mean of $\boldsymbol{x}$ that are not relevant here). By the Rayleigh-Ritz theorem (e.g., Horn and Johnson, 1985), $\boldsymbol{f}_{\max}$ represents a vector that is most autocorrelated according to the Moran statistic, and $\boldsymbol{f}_1(\boldsymbol{W}), ..., \boldsymbol{f}_{m_1}(\boldsymbol{W})$ represent vectors that are least autocorrelated. Note that λ_n, the value of the Moran static when $\boldsymbol{x} = \boldsymbol{f}_{\max}$, is positive by the Perron-Frobenius theorem, and λ_1, the value of the Moran static when $\boldsymbol{x} = \boldsymbol{f}_1(\boldsymbol{W}), ..., \boldsymbol{f}_{m_1}(\boldsymbol{W})$, is negative for $\mathrm{tr}(\boldsymbol{W}) = \sum_{i=1}^n \lambda_i(\boldsymbol{W}) = 0$ by assumption. We may thus conclude that in CAR and symmetric SAR models it is difficult to detect large positive spatial autocorrelation when the regressors are the sum of a strongly positively autocorrelated component and a strongly negatively autocorrelated component, with the former component being the dominant one.

Remark D.2 An extension of Proposition D.1 that is directly related to the above interpretation, and can be proved similarly to Proposition D.1, is worth mentioning. If $\boldsymbol{f}_j(\boldsymbol{W}) + \omega \boldsymbol{f}_{\max} \in \operatorname{col}(\boldsymbol{X})$, with $\boldsymbol{f}_j(\boldsymbol{W}) \neq \boldsymbol{f}_{\max}$ and $\boldsymbol{f}_j(\boldsymbol{W}) \notin E_{n-1}(\boldsymbol{W})$, the limiting power of a LBI test in a CAR or symmetric SAR model is 1 for any α (i.e., $\alpha^* = 0$) provided that $\omega < [\lambda_{\max} - \lambda_{n-1}(\boldsymbol{W})]^{1/2} [\lambda_{n-1}(\boldsymbol{W}) - \lambda_j(\boldsymbol{W})]^{-1/2}$. Such an expression can be used to infer how $\boldsymbol{W}$ affects (through its spectrum, under Gaussianity) the power properties of tests of $\rho = 0$. For instance, if $\boldsymbol{W}$ is such that $\lambda_{\max} - \lambda_{n-1}(\boldsymbol{W})$ is large, then any vector $\boldsymbol{X}$ in a large region of the plane spanned by $\boldsymbol{f}_j(\boldsymbol{W})$ and $\boldsymbol{f}_{\max}$ yields $\alpha^* = 0$.

Remark D.3 When $m_1 = 1$, as in the examples we are considering, and $\operatorname{col}(\boldsymbol{X})$ contains a vector $\boldsymbol{f}_1(\boldsymbol{W}) + \omega \boldsymbol{f}_{max}$ with large ω, the power function of a POI or LBI test goes to zero by Proposition D.1, but, in general, it does so very rapidly. This is because the condition $\boldsymbol{f}_{max} \in \operatorname{col}(\boldsymbol{X})$ is nearly satisfied, and hence the power function is close to that when $\boldsymbol{f}_{max} \in \operatorname{col}(\boldsymbol{X})$, which, by Theorem 3.3, goes to a positive number as $\rho \to \lambda_{\max}^{-1}$.

Appendix E The Condition in Proposition 3.10

The condition in Proposition 3.10 (i.e., $\boldsymbol{f}_{\max}$ is an eigenvector of $\boldsymbol{W}'$) is equivalent to $\lambda_{\max}$ being perfectly well-conditioned (Golub and Van Loan, 1996, p. 323), which is well-known to be a restrictive condition for a nonsymmetric matrix. It should be observed that, for any given choice of the neighborhood structure of a set of observational units (i.e., any choice of the pairs of units deemed to be neighbors), the non-zero elements of $\boldsymbol{W}$ can be taken in such a way that $\lambda_{\max}$ is well-conditioned. (This can be achieved by starting from a nonsymmetric matrix $\boldsymbol{W}$, and applying the similarity transformation $\boldsymbol{P}^{-1}\boldsymbol{W}\boldsymbol{P}$, where $\boldsymbol{P}$ is a diagonal matrix with $P_{i,i} = [(\boldsymbol{f}_{\max})_i/(\boldsymbol{l}_{\max})_i]^{1/2}$, and $\boldsymbol{l}_{\max}$ denotes the left eigenvector of $\boldsymbol{W}$ associated to $\lambda_{\max}$. Clearly, the resulting matrix has the same left and right eigenvector, and hence a well-conditioned eigenvalue of largest modulus.) However, in general, the choice of weights yielding a well-conditioned $\lambda_{\max}$ is a very particular one, and does not correspond to a relevant notion of distance amongst the observational units.

The restrictiveness of the condition that $\boldsymbol{f}_{\max}$ is an eigenvector of $\boldsymbol{W}'$ is very transparent in the important case of a row-standardized $\boldsymbol{W}$, as emphasized by the following corollary of Proposition 3.10.

Corollary E.1 *In a zero-mean SAR model with row-stochastic $\boldsymbol{W}$, the limiting power of a POI or LBI critical region for testing $\rho = 0$ against $\rho > 0$ is 1 for any α if and only if $\boldsymbol{W}$ is doubly stochastic.*

Proof. If $\boldsymbol{W}$ is a row-stochastic matrix, then $\boldsymbol{f}_{\max}$ has identical entries. Hence, the condition in Proposition 3.10 is satisfied if and only if the columns of $\boldsymbol{W}$, as its rows, sum to 1. ∎

Clearly, a nonsymmetric row-stochastic $\boldsymbol{W}$ is doubly stochastic, i.e., has not only all rows but also all columns summing to 1, only in very special circumstances. Formally, the restrictiveness of such circumstances can be deduced from Birkhoff's theorem on doubly stochastic matrices, which states that any such matrix must be a convex combination of permutation matrices (e.g., Horn and Johnson, 1985). We remark that the doubly stochastic weights matrices used by Pace and LeSage (2002) are not relevant here, because they are symmetric.

Appendix F Proofs and Remarks for Section 3

We first prove an auxiliary lemma, and then all other results in Section 3.

Lemma F.1 *Consider, in the context of CAR or symmetric SAR models, testing* $\rho = 0$ *against* $\rho > 0$ *by means of a critical region*

$$\Phi_{\boldsymbol{v}}(\boldsymbol{B}) = \{\boldsymbol{v} \in S_{n-k} : \boldsymbol{v}'\boldsymbol{B}\boldsymbol{v} < c_\alpha\},$$

where $\boldsymbol{B}$ *is an* $(n-k)\times(n-k)$ *known symmetric matrix that does not depend on* α*. Provided that* $\boldsymbol{f}_{\max} \notin \operatorname{col}(\boldsymbol{X})$*,* $\alpha^* = 0$ *if and only if* $\boldsymbol{C}\boldsymbol{f}_{\max} \in E_1(\boldsymbol{B})$*, and* $\alpha^* = 1$ *if and only if* $\boldsymbol{C}\boldsymbol{f}_{\max} \in E_{n-k}(\boldsymbol{B})$*.*

Proof. From (14), we have that, provided that $\boldsymbol{C}\boldsymbol{f}_{\max} \neq \boldsymbol{0}$, $\alpha^* = 0$ if and only if $\boldsymbol{C}\boldsymbol{f}_{\max}/\left\|\boldsymbol{C}\boldsymbol{f}_{\max}\right\| = \underset{\boldsymbol{v}\in S_{n-k}}{\arg\min}\{\boldsymbol{v}'\boldsymbol{B}\boldsymbol{v}\}$, and $\alpha^* = 1$ if and only if $\boldsymbol{C}\boldsymbol{f}_{\max}/\left\|\boldsymbol{C}\boldsymbol{f}_{\max}\right\| = \underset{\boldsymbol{v}\in S_{n-k}}{\arg\max}\{\boldsymbol{v}'\boldsymbol{B}\boldsymbol{v}\}$. The proposition follows by application of the Rayleigh-Ritz theorem (e.g., Horn and Johnson, 1985). ∎

Proof of Proposition 3.1. For some matrix $\boldsymbol{X}$, let $\Upsilon_{\boldsymbol{X}}(\bar\rho)$ be the size-α POI critical region, defined on the sample space. By definition, $\Upsilon_{\boldsymbol{X}}(\bar\rho)$ is the size-α critical region that is invariant under the group $F_{\boldsymbol{X}}$ (defined in Appendix B) and has maximum probability content under $\mathrm{N}(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{\Sigma}(\bar\rho))$. Observe that, by invariance under $F_{\boldsymbol{X}}$, the probability content $\pi_{\bar\rho}(\bar\rho;\boldsymbol{X})$ of $\Upsilon_{\boldsymbol{X}}(\bar\rho)$ under $\mathrm{N}(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{\Sigma}(\bar\rho))$ is the same as under $\mathrm{N}(\boldsymbol{0}, \boldsymbol{\Sigma}(\bar\rho))$, for any $\boldsymbol{X}$. Further, note that $F_{\boldsymbol{X}}$ is strictly larger than $F_{\mathbf{O}}$, because all transformations in $F_{\mathbf{O}}$, i.e. $\boldsymbol{y} \to a\boldsymbol{y}$, are in $F_{\boldsymbol{X}}$, and there are transformations in $F_{\boldsymbol{X}}$, i.e., those with $\boldsymbol{b} \neq \boldsymbol{0}$, that are not in $F_{\mathbf{O}}$. It follows that, for any $\boldsymbol{X} \neq \boldsymbol{0}$, any $\bar\rho > 0$, and any α, $\pi_{\bar\rho}(\bar\rho;\boldsymbol{X}) \leq \pi_{\bar\rho}(\bar\rho;\mathbf{O})$. Since, by the Neyman-Pearson Lemma applied to $pdf(\boldsymbol{v};\bar\rho)$, $\Upsilon_{\mathbf{O}}(\bar\rho)$ is unique up to a set of measure zero, a necessary and sufficient condition for $\pi_{\bar\rho}(\bar\rho;\boldsymbol{X}) = \pi_{\bar\rho}(\bar\rho;\mathbf{O})$, $\boldsymbol{X} \neq \mathbf{O}$, is that $\Upsilon_{\boldsymbol{X}}(\bar\rho) = \Upsilon_{\mathbf{O}}(\bar\rho)$. Such a condition is equivalent to $\boldsymbol{y}'\boldsymbol{C}'[\left(\boldsymbol{C}\boldsymbol{\Sigma}(\bar\rho)\boldsymbol{C}'\right)^{-1} - c_\alpha\boldsymbol{I}]\boldsymbol{C}\boldsymbol{y} < 0$ if and only if $\boldsymbol{y}'[\boldsymbol{\Sigma}^{-1}(\bar\rho) - c_\alpha\boldsymbol{I}]\boldsymbol{y} < 0$. Thus, since $\operatorname{rk}(\boldsymbol{C}'\boldsymbol{Q}\boldsymbol{C}) \leq n-k$ for any $(n-k)\times(n-k)$ matrix $\boldsymbol{Q}$, the condition $\Upsilon_{\boldsymbol{X}}(\bar\rho) = \Upsilon_{\mathbf{O}}(\bar\rho)$, $\boldsymbol{X} \neq \mathbf{O}$, requires $\operatorname{rk}(\boldsymbol{\Sigma}^{-1}(\bar\rho) - c_\alpha\boldsymbol{I}) \leq n-k$, and hence $c_\alpha = \lambda_i^{-1}(\boldsymbol{\Sigma}(\bar\rho))$, $i = 2,...,n-1$, or, equivalently, $\alpha = \Pr(\boldsymbol{y}'\boldsymbol{\Sigma}^{-1}(\bar\rho)\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{y} < \lambda_i^{-1}(\boldsymbol{\Sigma}(\bar\rho))$, $i = 2,...,n-2$ (the cases $i = 1$ and $i = n$ are excluded because α is assumed to be in $(0,1)$). It is now clear that if $c_\alpha = \lambda_i^{-1}(\boldsymbol{\Sigma}(\bar\rho))$, $i = 2,...,n-1$, then $\Upsilon_{\mathbf{O}}(\bar\rho)$ is invariant under $F_{\boldsymbol{X}}$, and hence $\Upsilon_{\boldsymbol{X}}(\bar\rho) = \Upsilon_{\mathbf{O}}(\bar\rho)$ if and only if $\operatorname{col}(\boldsymbol{X}) \subseteq E_i(\boldsymbol{\Sigma}(\rho))$, completing the proof. ∎

Proof of Lemma 3.2. Let $p(\boldsymbol{v}) := \lim_{\rho\to a} pdf(\boldsymbol{v};\rho) = 2\left|\boldsymbol{\Omega}\right|^{-\frac{1}{2}}\left(\boldsymbol{v}'\boldsymbol{\Omega}^{-1}\boldsymbol{v}\right)^{-(n-k)/2}$. If all the eigenvalues of $\boldsymbol{\Sigma}(\rho)$ tend to a positive value as $\rho \to a$, then, by the Poincaré separation theorem (e.g., Horn and Johnson, 1985), all the eigenvalues of $\boldsymbol{\Omega}$ are positive. It follows that the term $\left|\boldsymbol{\Omega}\right|^{-\frac{1}{2}}$ of $p(\boldsymbol{v})$ is positive and finite if $\lambda_{n-k}(\boldsymbol{\Omega}) < \infty$, and it vanishes otherwise. Let now $\widetilde{E}_{n-k}(\boldsymbol{\Omega}_\rho) = S_{n-k} \cap E_{n-k}(\boldsymbol{\Omega}_\rho)$. The term $(\boldsymbol{v}'\boldsymbol{\Omega}^{-1}\boldsymbol{v})^{-(n-k)/2}$ of $p(\boldsymbol{v})$ is infinite if $\lambda_{n-k}(\boldsymbol{\Omega}) = \infty$ and $\boldsymbol{v} \in \widetilde{E}_{n-k}(\boldsymbol{\Omega})$, positive and finite in any other case. Combining the results, we have that if $\lambda_{n-k}(\boldsymbol{\Omega}) = \infty$, then $p(\boldsymbol{v}) = 0$ for any $\boldsymbol{v} \notin \widetilde{E}_{n-k}(\boldsymbol{\Omega})$. Observe that, since it is the intersection between an hemisphere and the subspace $E_{n-k}(\boldsymbol{\Omega}_\rho)$ containing the centre of the hemisphere, $\widetilde{E}_{n-k}(\boldsymbol{\Omega}_\rho)$ is a singleton if and only if $E_{n-k}(\boldsymbol{\Omega}_\rho)$ is one-dimensional. It follows that (i) if $\lambda_{n-k}(\boldsymbol{\Omega})$ is infinite and has algebraic multiplicity one, then $p(\boldsymbol{v})$ must be infinite when $\boldsymbol{v} \in \widetilde{E}_{n-k}(\boldsymbol{\Omega})$; (ii) if $\lambda_{n-k}(\boldsymbol{\Omega}) < \infty$, then $0 < p(\boldsymbol{v}) < \infty$. The proof is now completed, because the power of a critical region as $\rho \to a$ is the probability content of the region under $p(\boldsymbol{v})$. ∎

Proof of Theorem 3.3. For any weights matrix $\boldsymbol{W}$ and any $\rho > 0$, the matrix $(\boldsymbol{I} - \rho\boldsymbol{W})^{-1}$ is entrywise positive (see, e.g., Gantmacher 1974, p. 69, and recall that we are implicitly assuming $\rho < \lambda_{\max}^{-1}$). Thus, by Perron's theorem (Horn and Johnson, 1985, Theorem 8.2.11), $\lambda_n[\boldsymbol{\Sigma}(\rho)]$ has algebraic multiplicity one for any $\rho > 0$ and any CAR and SAR model. Also, observe that as $\rho \to \lambda_{\max}^{-1}$, $\lambda_n[\boldsymbol{\Sigma}(\rho)] \to \infty$ and all other eigenvalues of $\boldsymbol{\Sigma}(\rho)$ tend to a finite value, because, as it is easily verified, $\mathrm{rk}[(\boldsymbol{I} - \lambda_{\max}^{-1}\boldsymbol{W}^{'})(\boldsymbol{I} - \lambda_{\max}^{-1}\boldsymbol{W})] = n - 1$. For CAR and symmetric SAR models, the spectral decomposition $\boldsymbol{\Sigma}(\rho) = \sum_{i=1}^{n}\{\lambda_i[\boldsymbol{\Sigma}(\rho)]\boldsymbol{f}_i[\boldsymbol{\Sigma}(\rho)]\boldsymbol{f}_i'[\boldsymbol{\Sigma}(\rho)]\}$ immediately shows that the matrix $\lambda_n^{-1}[\boldsymbol{\Sigma}(\rho)]\,\boldsymbol{\Omega}_\rho$ tends to $\boldsymbol{C}\boldsymbol{f}_{\max}\boldsymbol{f}_{\max}'\boldsymbol{C}'$ as $\rho \to \lambda_{\max}^{-1}$, because for those models $\boldsymbol{f}_n[\boldsymbol{\Sigma}(\rho)] = \boldsymbol{f}_{\max}$. The same limit result holds also for asymmetric SAR models, since in that case $\boldsymbol{f}_n[\boldsymbol{\Sigma}(\rho)] \to \boldsymbol{f}_{\max}$ as $\rho \to \lambda_{\max}^{-1}$ (because $(\boldsymbol{I} - \lambda_{\max}^{-1}\boldsymbol{W}')(\boldsymbol{I} - \lambda_{\max}^{-1}\boldsymbol{W})$ has the eigenvector $\boldsymbol{f}_{\max}$ corresponding to its smallest eigenvalue 0). Now, since $\mathrm{rk}(\boldsymbol{C}\boldsymbol{f}_{\max}\boldsymbol{f}_{\max}'\boldsymbol{C}') \leq \mathrm{rk}(\boldsymbol{f}_{\max}\boldsymbol{f}_{\max}') = 1$, all eigenvalues of $\boldsymbol{C}\boldsymbol{f}_{\max}\boldsymbol{f}_{\max}'\boldsymbol{C}'$ are zero except possibly one, which must then be equal to $\bar{\lambda} = \mathrm{tr}(\boldsymbol{C}\boldsymbol{f}_{\max}\boldsymbol{f}_{\max}'\boldsymbol{C}') = \boldsymbol{f}_{\max}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{f}_{\max}$. If $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$, then $\bar{\lambda}$ is a positive eigenvalue of $\boldsymbol{C}\boldsymbol{f}_{\max}\boldsymbol{f}_{\max}'\boldsymbol{C}'$ with algebraic multiplicity 1 and has an associated eigenvector equal to $\boldsymbol{C}\boldsymbol{f}_{\max}$, for

$$\boldsymbol{C}\boldsymbol{f}_{\max}\boldsymbol{f}_{\max}'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{f}_{\max} = \boldsymbol{C}\boldsymbol{f}_{\max}\boldsymbol{f}_{\max}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{f}_{\max} = \bar{\lambda}\boldsymbol{C}\boldsymbol{f}_{\max}.$$

Because of the continuity of the eigenvalues of a matrix in the entries of that matrix, it is then easily seen that for any CAR or SAR model such that $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$, $\boldsymbol{C}\boldsymbol{f}_{\max}$ span an eigenspace of $\lim\limits_{\rho\to\lambda_{\max}^{-1}} \boldsymbol{\Omega}_\rho$ associated to the eigenvalue $\lim\limits_{\rho\to\lambda_{\max}^{-1}} \{\lambda_n\,[\boldsymbol{\Sigma}(\rho)]\,\bar{\lambda}\} = \infty$. If $\boldsymbol{f}_{\max} \in \mathrm{col}(\boldsymbol{X})$, then $\boldsymbol{C}\boldsymbol{f}_{\max} = \boldsymbol{0}$ and thus $\boldsymbol{\Omega}_\rho = \sum_{i=1}^{n-1}\{\lambda_i^{-1}[\boldsymbol{\Sigma}(\rho)]\boldsymbol{C}\boldsymbol{f}_i(\boldsymbol{W})\boldsymbol{f}_i'(\boldsymbol{W})\boldsymbol{C}'\}$, which tends to a matrix whose entries are all finite. Hence, when $\boldsymbol{f}_{\max} \in \mathrm{col}(\boldsymbol{X})$, $\lim\limits_{\rho\to\lambda_{\max}^{-1}} \lambda_{n-k}\,(\boldsymbol{\Omega}_\rho)$ must be finite. The theorem now follows by applying Lemma 3.2 with $a = \lambda_{\max}^{-1}$. ∎

Remark F.2 The condition $\boldsymbol{f}_{\max} \in \mathrm{col}(\boldsymbol{X})$ in Theorem 3.3 is satisfied whenever $\boldsymbol{W}$ in a CAR or SAR model is row-standardized and an intercept is included in the regression, because row-standardization implies that $\boldsymbol{f}_{\max}$ has identical entries. Note that here whether $\boldsymbol{W}$ refers to a model before or after normalization to $\boldsymbol{\Sigma}(0) = \boldsymbol{I}$ (see Section 2.1) is irrelevant, because the condition $\boldsymbol{f}_{\max} \in \mathrm{col}(\boldsymbol{X})$ is invariant under any invertible linear transformation of $\boldsymbol{y} \sim \mathrm{N}(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{\Sigma}(\rho))$, where $\boldsymbol{\Sigma}(\rho)$ is that of a CAR or SAR model. For any weights matrix that is not row-standardized, in general $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$, with the consequence that the limiting power of an invariant test is either 0 or 1.

Remark F.3 An important and immediate consequence of Theorem 3.3 is that, in CAR and SAR models, the limit of the envelope $\pi_\rho(\rho)$ as $\rho \to \lambda_{\max}^{-1}$ is 1 if $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$, and is in $(\alpha, 1)$ otherwise. Hence, as $\rho \to \lambda_{\max}^{-1}$ in CAR and SAR models, the null hypothesis $\rho = 0$ can be distinguished (by means of an invariant tests) from the alternative hypothesis $\rho > 0$ with zero type II error probability only if $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$.

Remark F.4 By Theorem 3.3, whether the limiting power of a certain invariant test is 0, 1, or in $(0, 1)$ depends on the weights matrix that has generated the data only through $\boldsymbol{f}_{\max}$, and does not depend on whether the model is a CAR or a SAR model. This property implies some robustness of invariant tests, when spatial autocorrelation is large. For example, the limiting power of a certain invariant critical region is the same for all row-standardized $\boldsymbol{W}$'s appearing in a CAR or SAR model, because all such matrices have the same $\boldsymbol{f}_{\max}$.

Proof of Lemma 3.5. Assume $\boldsymbol{f}_{\max} \notin \operatorname{col}(\boldsymbol{X})$. By Theorem 3.3, the limiting power of a critical region in form (11) is 1 if $T(\boldsymbol{f}_{\max}) < c_\alpha$. This is equivalent to stating that α^* is the probability that $T(\boldsymbol{y}) < T(\boldsymbol{f}_{\max})$ under the null hypothesis $\boldsymbol{y} \sim \mathrm{N}(\boldsymbol{X\beta}, \sigma^2\boldsymbol{I})$, or, by invariance, under $\boldsymbol{y} \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{I})$. ∎

Proof of Theorem 3.6. We start from the case of the LBI test, which is slightly simpler than the case of POI tests. By Lemma F.1, for CAR and symmetric SAR models, the limiting power of a LBI test vanishes for any α if and only if $\boldsymbol{f}_{\max} \notin \operatorname{col}(\boldsymbol{X})$ and $\boldsymbol{Cf}_{\max} \in E_1(\boldsymbol{CWC}')$. We first consider the case $k = m_1$. For a fixed $\boldsymbol{W}$, consider the m_1-dimensional subspaces belonging to the span of $\boldsymbol{f}_1(\boldsymbol{W}), ..., \boldsymbol{f}_{m_1}(\boldsymbol{W}), \boldsymbol{f}_{\max}$, and denote by Θ the set of all such subspaces that do not contain $\boldsymbol{f}_{\max}$ and are not $E_1(\boldsymbol{W})$. It can be easily shown that, if $\operatorname{col}(\boldsymbol{X}) \in \Theta$, $\boldsymbol{CWC}'$ admits the eigenpairs $(\lambda_i(\boldsymbol{W}), \boldsymbol{Cf}_i(\boldsymbol{W}))$, $i = m_1 + 1, ..., n-1$. But then, by the symmetry of $\boldsymbol{CWC}'$ and the fact that the vectors $\boldsymbol{Cf}_i(\boldsymbol{W})$, $i = m_1 + 1, ..., n-1$ are pairwise orthogonal (because the $\boldsymbol{f}_i(\boldsymbol{W})$'s are), $\boldsymbol{CWC}'$ must also admit an eigenvector in the subspace spanned by $\boldsymbol{Cf}_1(\boldsymbol{W}), ..., \boldsymbol{Cf}_{m_1}(\boldsymbol{W}), \boldsymbol{Cf}_{\max}$. Since when $\operatorname{col}(\boldsymbol{X}) \in \Theta$ such a subspace is 1-dimensional, it follows that $\boldsymbol{Cf}_{\max}$ is an eigenvector of $\boldsymbol{CWC}'$, i.e.,

$$\boldsymbol{CWM_X f}_{\max} = \tilde{\lambda}\boldsymbol{Cf}_{\max} \tag{15}$$

for some eigenvalue $\tilde{\lambda}$. Thus, a $\operatorname{col}(\boldsymbol{X}) \in \Theta$ causing the limiting power of the LBI test to disappear for any α exists if and only if $\tilde{\lambda} \leq \lambda_{m_1+1}(\boldsymbol{W})$. Observe that as $\operatorname{col}(\boldsymbol{X}) \in \Theta$ approaches a subspace orthogonal to $E_1(\boldsymbol{W})$, $\boldsymbol{M_X f}_{\max}/\left\|\boldsymbol{M_X f}_{\max}\right\|$ tends to a vector in $E_1(\boldsymbol{W})$, which implies that $\tilde{\lambda} \to \lambda_1(\boldsymbol{W})$ (note that, by the definition of Θ, no $\operatorname{col}(\boldsymbol{X}) \in \Theta$ is orthogonal to $E_1(\boldsymbol{W})$). Hence, by the continuity of the eigenvalues of a matrix ($\boldsymbol{CWC}'$ here) in the entries of the matrix itself plus the fact that $\lambda_1(\boldsymbol{W}) < \lambda_{m_1+1}(\boldsymbol{W})$, a $\operatorname{col}(\boldsymbol{X}) \in \Theta$ such that $\tilde{\lambda} \leq \lambda_{m_1+1}(\boldsymbol{W})$ always exists. The extension to POI tests, for any $\bar{\rho} > 0$, is straightforward and is obtained by replacing $\boldsymbol{W}$ with $\boldsymbol{\Sigma}(\bar{\rho})$ and λ_i by $(1 - \rho\lambda_i)^{-r}$, $i = 1, ..., n$, in the arguments used above, for any CAR ($r = 1$) or symmetric SAR model ($r = 2$). We have thus proved the proposition for the case $k = m_1$. But, since the power of an invariant test does not depend on $\boldsymbol{\beta}$, it easily follows that, if an m_1-dimensional regression space causes a zero limiting power (of a POI or LBI test in a CAR or symmetric SAR model), then also all the k-dimensional regression spaces, with $k > m_1$, that contain it but do not contain $\boldsymbol{f}_{\max}$ yield a zero limiting power. The proof is therefore completed. ∎

Remark F.5 The strongest implication of Theorem 3.6 is perhaps that regression spaces such that the limiting power of a POI test vanishes can be found even when $\bar{\rho}$ is large (i.e., close to $\lambda_{\max}^{-1}$) and α is large. This is surprising because, by the Neyman-Pearson Lemma, the power at $\bar{\rho}$ of a POI test must be larger than α. Since, if $\boldsymbol{f}_{\max} \notin \operatorname{col}(\boldsymbol{X})$, $\pi_\rho(\rho) \to 1$ as $\rho \to \lambda_{\max}^{-1}$ (see Remark F.3), it also holds that the supremum—as $\operatorname{col}(\boldsymbol{X})$ ranges over the set of all k-dimensional, $k \geq m_1$, subspaces of $\mathbb{R}^n$—of the maximum shortcoming (e.g., Lehmann and Romano, 2005, p. 337) of any POI or LBI test is always one, for any $\boldsymbol{W}$ and any α.

Remark F.6 For any POI or LBI test and any $\boldsymbol{W}$, $T(\boldsymbol{f}_{\max})$, regarded as a function from $G_{k,n}$ to $\mathbb{R}$, is continuous. Thus, by (14), α^* is itself a continuous, and generally smooth, function of $\operatorname{col}(\boldsymbol{X})$. It follows, in particular, that the regression spaces that are close (according to some distance on $G_{k,n}$) to regression spaces yielding a large (resp. small) α^* yield a large (resp. small) α^*.

Remark F.7 We have not attempted to generalize Theorem 3.6 to asymmetric SAR models, due to two reasons. Firstly, such models generally do not satisfy the condition $\boldsymbol{f}_{\max} \notin \operatorname{col}(\boldsymbol{X})$ necessary for the zero limiting power problem. This is because the nonsymmetric weights matrices generally used in SAR models are row-stochastic, implying, as already noted above, that $\boldsymbol{f}_{\max} \in \operatorname{col}(\boldsymbol{X})$ as long as an intercept is included in the regression. Secondly, although the proof of Theorem 3.6 suggests that regression spaces (of low dimension) such that the limiting power of a POI or LBI test vanishes for any α always exist also in the context of asymmetric SAR models, the exact characterization of such regression spaces appears to be more involved. It should be noted, however, that an approximated characterization can be obtained from Theorem 3.6, by approximating an asymmetric SAR model by a CAR model with $\boldsymbol{\Sigma}^{-1}(\rho) = \boldsymbol{I} - \rho(\boldsymbol{W} + \boldsymbol{W}')$ (i.e., omitting terms in ρ^2).

Proof of Theorem 3.8. By Theorem 3.6, there exists at least one subspace $\operatorname{col}(\boldsymbol{X}) \in G_{k,n}$, $k \geq m_1$, such that, for a POI or LBI test in a CAR or symmetric SAR model, $\alpha^* = 1$. Let $T(\boldsymbol{y})$ represents the test statistic associated to a POI or LBI test. Then, by Lemma 3.5, the subspaces $\operatorname{col}(\boldsymbol{X})$ yielding $\alpha^* = 1$ are those that maximize $T(\boldsymbol{f}_{\max})$, regarded as a function from $G_{k,n}$ to $\mathbb{R}$. Since $T(\boldsymbol{f}_{\max})$ is continuous at its points of maximum, it follows that, for any α, it is possible to find a neighborhood (defined according to some arbitrary distance on $G_{k,n}$) of the points of maximum such that any $\operatorname{col}(\boldsymbol{X})$ in this neighborhood causes the limiting power of size-α tests to disappear. This implies immediately that $H_k(\alpha)$ has non-zero invariant measure on $G_{k,n}$ (see James, 1954), for any α and for $k = m_1$, and for any POI or LBI tests in any CAR or symmetric SAR model. Since the power of an invariant test does not depend on $\boldsymbol{\beta}$, the result also holds for $k > m_1$. ∎

Proof of Corollary 3.9. The result follows immediately by taking $\boldsymbol{C} = \boldsymbol{I}$ in Theorem 3.3. ∎

Proof of Proposition 3.10. When $\boldsymbol{W}$ is symmetric, i.e., in the case of CAR or symmetric SAR models, and $\boldsymbol{X} = \mathbf{O}$ the eigenspace associated to the smallest eigenvalue of both matrices in the middle of the quadratic forms in (8) and (9) is spanned by $\boldsymbol{f}_{\max}$. Thus, when $\boldsymbol{W}$ is symmetric and $\boldsymbol{X} = \mathbf{O}$, and for a POI or LBI test, $\alpha^* = 0$ by Lemma F.1. The part of the proposition relative to asymmetric SAR models requires a little more work. Observe that if $\boldsymbol{f}_{\max}$ is an eigenvector of $\boldsymbol{W}'$, it must be associated to $\lambda_{\max}^{-1}$. To see this, call ϕ the eigenvalue of $\boldsymbol{W}'$ associated to $\boldsymbol{f}_{\max}$. Transposing both left and right hand sides of the equation $\boldsymbol{W}'\boldsymbol{f}_{\max} = \phi \boldsymbol{f}_{\max}$ and post-multiplying them by $\boldsymbol{f}_{\max}$ yield $\boldsymbol{f}'_{\max}\boldsymbol{W}\boldsymbol{f}_{\max} = \phi$. But then $\phi = \lambda_{\max}$, because it must also hold that $\boldsymbol{f}'_{\max}\boldsymbol{W}\boldsymbol{f}_{\max} = \lambda_{\max}$. Let $\boldsymbol{\Gamma}(\rho) = [(\boldsymbol{I} - \rho\boldsymbol{W}')(\boldsymbol{I} - \rho\boldsymbol{W})]^{-1}$. By Lemma F.1 with $\boldsymbol{B} = \boldsymbol{\Sigma}^{-1}(\bar{\rho})$, to prove the part of the proposition regarding POI tests in asymmetric SAR models we need to show that $\boldsymbol{W}'\boldsymbol{f}_{\max} = \lambda_{\max}\boldsymbol{f}_{\max}$ is necessary and sufficient for $\boldsymbol{f}_{\max} \in E_n(\boldsymbol{\Gamma}(\bar{\rho}))$, for any $\bar{\rho} > 0$. Clearly, if this holds for any $\bar{\rho} > 0$, it also holds for $\bar{\rho} \to 0$, establishing the part of the proposition regarding the LBI test in asymmetric SAR models. The necessity is straightforward, because if $\boldsymbol{\Gamma}(\bar{\rho})\boldsymbol{f}_{\max} = \lambda_n(\boldsymbol{\Gamma}(\bar{\rho}))\boldsymbol{f}_{\max}$, then $\boldsymbol{\Gamma}^{-1}(\bar{\rho})\boldsymbol{f}_{\max} = \lambda_n^{-1}(\boldsymbol{\Gamma}(\bar{\rho}))\boldsymbol{f}_{\max}$. From the latter equation we have $(1 - \bar{\rho}\lambda_{\max})(\boldsymbol{I} - \rho\boldsymbol{W}')\boldsymbol{f}_{\max} = \lambda_{\max}^{-1}(\boldsymbol{\Gamma}(\bar{\rho}))\boldsymbol{f}_{\max}$, which requires $\boldsymbol{f}_{\max}$ to be an eigenvector of $\boldsymbol{I} - \rho\boldsymbol{W}'$ and hence of $\boldsymbol{W}'$ (associated to $\lambda_{\max}$ by the above argument). As for the sufficiency, note that if $\boldsymbol{W}'\boldsymbol{f}_{\max} = \lambda_{\max}\boldsymbol{f}_{\max}$, then $\boldsymbol{f}_{\max}$ is clearly an eigenvector of $\boldsymbol{\Gamma}(\bar{\rho})$, for any $\bar{\rho} > 0$. By Perron's theorem (e.g., Horn and Johnson, 1985, Theorem 8.2.11), a vector in $E_n(\boldsymbol{\Gamma}(\bar{\rho}))$ is entrywise nonnegative (or nonpositive), for any $\bar{\rho} > 0$. But, by the Perron-Frobenius theorem applied to $\boldsymbol{W}$, $\boldsymbol{f}_{\max}$ is entrywise positive. Hence, $\boldsymbol{f}_{\max}$ must be in $E_n(\boldsymbol{\Gamma}(\bar{\rho}))$, for any $\bar{\rho} > 0$, because, since $\boldsymbol{\Gamma}(\bar{\rho})$ is symmetric, if it were not, then it should be orthogonal to an entrywise nonnegative vector, which is impossible. This completes the proof of the proposition. ∎

Appendix G Proofs and Remarks for Section 4

We first prove an auxiliary lemma, and then all other results in Section 4.

Lemma G.1 *Consider, in the context of CAR or symmetric SAR models, testing $\rho = 0$ against $\rho > 0$ by means of a POI or LBI test. Provided that $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$, $\alpha^* = 0$ if $E_{n-k}(\boldsymbol{\Omega}_\rho)$ does not depend on ρ for $\rho > 0$.*

Proof. It can be deduced from the proof of Theorem 3.3 that, for a CAR or SAR model with $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$, the limiting, as $\rho \to \lambda_{\max}^{-1}$, eigenspace $E_{n-k}(\boldsymbol{\Omega}_\rho)$ is 1-dimensional and contains $\boldsymbol{C}\boldsymbol{f}_{\max}$. It follows that if $E_{n-k}(\boldsymbol{\Omega}_\rho)$ does not depend on ρ for $\rho > 0$, it must be spanned by $\boldsymbol{C}\boldsymbol{f}_{\max}$ for any $\rho > 0$. Thus, for a POI test, $\alpha^* = 0$ by application of Lemma F.1 with $\boldsymbol{B} = \boldsymbol{\Omega}_{\bar{\rho}}^{-1}$. Since this property holds for any $\bar{\rho} > 0$, it also holds for the LBI test. ∎

Remark G.2 Two particular cases that are easily seen to satisfy the condition in Lemma G.1 are: (i) $\boldsymbol{W}$ symmetric and $\boldsymbol{X} = \mathbf{O}$; (ii) $\boldsymbol{W}$ symmetric and $\boldsymbol{f}_{\max} \perp \mathrm{col}(\boldsymbol{X})$.

Proof of Proposition 4.1. By Lemma 2 of King (1980), $\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{\Sigma}(\bar{\rho})\boldsymbol{C}')^{-1}\boldsymbol{C} = \boldsymbol{\Sigma}^{-1}(\bar{\rho})\boldsymbol{R}(\bar{\rho})$, where $\boldsymbol{R}(\bar{\rho}) := \boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}(\bar{\rho})\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}(\bar{\rho})$. Then, for any $0 \leq \rho < \lambda_{\max}^{-1}$, any $\bar{\rho}$, and any α, the power of the POI critical region (8) can be written as

$$\pi_{\bar{\rho}}(\rho) = \Pr\left(\frac{\boldsymbol{y}'\boldsymbol{\Sigma}^{-1}(\bar{\rho})\boldsymbol{R}(\bar{\rho})\boldsymbol{y}}{\boldsymbol{y}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{y}} < c_\alpha;\ \boldsymbol{y} \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Sigma}(\rho))\right), \tag{16}$$

where c_α is a constant such that the size of the test is α. Under Conditions A and B it is easily seen that $\boldsymbol{R}(\bar{\rho}) = \boldsymbol{M}_{\boldsymbol{X}}$ and $\boldsymbol{\Sigma}(\bar{\rho})\boldsymbol{X} = \boldsymbol{X}\boldsymbol{A}$, for any $\bar{\rho} > 0$. It follows that, under Conditions A and B, the matrices $\boldsymbol{\Sigma}^{-1}(\bar{\rho})$ and $\boldsymbol{M}_{\boldsymbol{X}}$ commute for any $\bar{\rho} > 0$, and hence

$$\pi_{\bar{\rho}}(\rho) = \Pr\left(\frac{\boldsymbol{z}'\boldsymbol{\Sigma}(\rho)\boldsymbol{\Sigma}^{-1}(\bar{\rho})\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{z}}{\boldsymbol{z}'\boldsymbol{\Sigma}(\rho)\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{z}} < c_\alpha\right),$$

where $\boldsymbol{z} \sim \mathrm{N}(\mathbf{0}, \boldsymbol{I})$. Moreover, under the same conditions, the matrix $\boldsymbol{M}_{\boldsymbol{X}}$ has an eigenvalue 0 with eigenspace spanned by the k eigenvectors of $\boldsymbol{\Sigma}(\rho)$ that are in $\mathrm{col}(\boldsymbol{X})$, and an eigenvalue 1 with eigenspace spanned by the remaining eigenvectors of $\boldsymbol{\Sigma}(\rho)$. Let H be the set of indexes i of the $n-k$ eigenvalues $\lambda_i[\boldsymbol{\Sigma}(\rho)]$ associated to a set of linearly independent eigenvectors of $\boldsymbol{\Sigma}(\rho)$ that are not in $\mathrm{col}(\boldsymbol{X})$. Note that, when Condition A holds, H does not depend on ρ. Under Conditions A and B, the power of a POI critical region can then be expressed as

$$\pi_{\bar{\rho}}(\rho) = \Pr\left(\frac{\sum_{i \in H}\lambda_i[\boldsymbol{\Sigma}(\rho)]\lambda_i^{-1}[\boldsymbol{\Sigma}(\bar{\rho})]z_i^2}{\sum_{i \in H}\lambda_i[\boldsymbol{\Sigma}(\rho)]z_i^2} < c_\alpha\right), \tag{17}$$

and its size as

$$\alpha = \Pr\left(\frac{\boldsymbol{z}'\boldsymbol{\Sigma}^{-1}(\bar{\rho})\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{z}}{\boldsymbol{z}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{z}} < c_\alpha\right) = \Pr\left(\frac{\sum_{i \in H}\lambda_i^{-1}[\boldsymbol{\Sigma}(\bar{\rho})]z_i^2}{\sum_{i \in H}z_i^2} < c_\alpha\right). \tag{18}$$

Observe now that the sequences $\lambda_i[\boldsymbol{\Sigma}(\rho)]$, $i \in H$, and $\lambda_i^{-1}[\boldsymbol{\Sigma}(\bar{\rho})]$, $i \in H$, are oppositely ordered in the sense of Hardy *et al.* (1952) p. 43. Then, the application of Tchebychef's inequality (Hardy *et al.*, 1952, Theorem 43) to the weighted arithmetic means (with weights $z_i^2/\sum_{i \in H}z_i^2$) of the $\lambda_i[\boldsymbol{\Sigma}(\rho)]$, $i \in H$, and of the $\lambda_i^{-1}[\boldsymbol{\Sigma}(\bar{\rho})]$, $i \in H$, yields that

$$\sum_{i \in H}\lambda_i[\boldsymbol{\Sigma}(\rho)]z_i^2 \sum_{i \in H}\lambda_i^{-1}[\boldsymbol{\Sigma}(\bar{\rho})]z_i^2 \geq \sum_{i \in H}z_i^2 \sum_{i \in H}\lambda_i[\boldsymbol{\Sigma}(\rho)]\lambda_i^{-1}[\boldsymbol{\Sigma}((\bar{\rho})]z_i^2,$$

for any vector $\boldsymbol{z} \in \mathbb{R}^n$, with equality holding only if all the $\lambda_i[\boldsymbol{\Sigma}(\rho)]$ or all the $\lambda_i^{-1}[\boldsymbol{\Sigma}((\bar{\rho})]$, $i \in H$, are the same. Rearranging the terms of the above inequality, one immediately sees that the statistic appearing in expression (17) is stochastically larger (e.g., Lehmann and Romano, 2005, p. 70) than that appearing in expression (18), and hence that $\pi_{\bar{\rho}}(\rho) \geq \alpha$, for any $\bar{\rho} > 0$, any $\rho > 0$ and any size α. If there are at least two indexes $i, j \in H$ such that $\lambda_i[\boldsymbol{\Sigma}(\rho)] \neq \lambda_j[\boldsymbol{\Sigma}(\rho)]$, i.e., if $\mathrm{col}^{\perp}(\boldsymbol{X})$ is not a subset of an eigenspace of $\boldsymbol{\Sigma}(\rho)$, then the last inequality is strict (as we are assuming $\alpha \neq 0, 1$). We have therefore proved the part of the proposition relative to POI tests. But, if the proposition holds for any POI test, i.e. for any $\bar{\rho}$, then it clearly also holds for the LBI test. ∎

Remark G.3 By the Neyman-Pearson lemma, $\pi_\rho(\rho) > \alpha$ for any $\rho > 0$ (this follows, for instance, by applying Corollary 3.2.1 of Lehmann and Romano (2005) to the density (7), plus the assumption of identification of model $\mathrm{N}(\boldsymbol{X\beta}, \sigma^2\boldsymbol{\Sigma}(\rho))$, or by applying the indirect argument in the proof of Theorem 1 in Kadiyala (1970)). Thus, if a UMPI test exists, the POI and LBI critical regions for the testing problem (6) are strictly unbiased. In general, Conditions A and B are not sufficient for the existence of a UMPI test, and therefore Proposition 4.1 is not a consequence of the above inequality. An important case in which a UMPI exists is a CAR model satisfying Condition B (in fact, Condition B combines particularly well with a CAR specification because the resulting model is an exponential family with number of sufficient statistics equal to the number of parameters, $k + 2$).

Proof of Proposition 4.2. For a CAR or a symmetric SAR model, $\lambda_i[\boldsymbol{\Sigma}(\rho)] = [1 - \rho\lambda_i(\boldsymbol{W})]^{-r}$, for $i = 1, ..., n$, and with $r = 1$ for a CAR model, $r = 2$ for a symmetric SAR model. Inserting such expressions in equation (17) from the proof of Proposition 4.1, we obtain that the power function of a POI critical region is non-decreasing in ρ if the statistic

$$t_{\bar{\rho}}(\rho) = \left\{ \sum_{i \in H} \left[\frac{1}{1 - \rho\lambda_i(\boldsymbol{W})} \right]^r z_i^2 \right\}^{-1} \sum_{i \in H} \left[\frac{1 - \bar{\rho}\lambda_i(\boldsymbol{W})}{1 - \rho\lambda_i(\boldsymbol{W})} \right]^r z_i^2$$

is non-increasing in ρ for any vector $\boldsymbol{z} \in \mathbb{R}^n$. Direct differentiation of $t_{\bar{\rho}}(\rho)$ with respect to ρ and some simple manipulation show that such a condition is satisfied if

$$\sum_{i,j \in H} a_{i,j} z_i^2 z_j^2 \leq 0, \tag{19}$$

with the coefficients $a_{i,j}$ defined by

$$a_{i,j} = \lambda_j \frac{[1 - \bar{\rho}\lambda_j(\boldsymbol{W})]^r - [1 - \bar{\rho}\lambda_i(\boldsymbol{W})]^r}{[1 - \rho\lambda_i(\boldsymbol{W})]^r [1 - \rho\lambda_j(\boldsymbol{W})]^{r+1}}.$$

It is immediately verified that, for each $i, j \in H$ such that $i \neq j$, $a_{i,j} + a_{j,i} \leq 0$, with strict inequality if $\lambda_i(\boldsymbol{W}) \neq \lambda_j(\boldsymbol{W})$. Thus, given that $a_{i,i} = 0$ for any $i \in H$, (19) holds, the inequality being strict if there exist at least one pair of distinct eigenvalues $\lambda_i(\boldsymbol{W}), \lambda_j(\boldsymbol{W})$ with $i, j \in H$, i.e., if $\mathrm{col}^{\perp}(\boldsymbol{X})$ is not a subset of an eigenspace of $\boldsymbol{W}$. The statement in the proposition relative to the POI critical regions is therefore proved, and the one relative to LBI follows immediately. ∎

Remark G.4 For CAR models, Proposition 4.2 can alternatively be proved by showing that the density $\mathit{pdf}(\boldsymbol{v}; \rho)$ has a monotone likelihood ratio under Condition B, and then by using Theorem 3.4.1 of Lehmann and Romano (2005). Such an argument, however, does not extend to symmetric SAR models.

Remark G.5 The power functions in Proposition 4.2 are, in fact, typically strictly increasing, because, unless $n-k$ is small or an eigenspace of $\boldsymbol{W}$ has large dimension below), $\mathrm{col}^{\perp}(\boldsymbol{X})$ is unlikely to fall into an eigenspace of $\boldsymbol{W}$. In the special case $\boldsymbol{X} = \mathbf{O}$, the power functions must be strictly increasing, for $\mathrm{col}^{\perp}(\boldsymbol{X}) = \mathbb{R}^n$ is certainly not in an eigenspace of $\boldsymbol{W}$.

Remark G.6 One case in which Condition B is satisfied is a CAR model with mean assumed to be unknown but constant across observations and with a row-standardized weights matrix. (On setting $\boldsymbol{L} = \boldsymbol{D}^{-1}$ and normalizing to $\boldsymbol{\Sigma}(0) = \boldsymbol{I}$, the mean of the CAR model becomes proportional to $\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\iota}$, and the covariance matrix becomes $\boldsymbol{\Sigma}(\rho) = \sigma^2(\boldsymbol{I} - \rho\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{D}^{-\frac{1}{2}})^{-1}$. Since $\boldsymbol{\iota}$ is an eigenvector of $\boldsymbol{D}^{-1}\boldsymbol{A}$, it follows that $\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\iota}$ is an eigenvector of $\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{D}^{-\frac{1}{2}}$ and hence of $\boldsymbol{\Sigma}(\rho)$, implying that Condition B is satisfied.) When other regressors are included alongside the intercept, a case in which Condition B has some chances of being met in practice is when the number of eigenspaces of $\boldsymbol{W}$ (and hence of $\boldsymbol{\Sigma}(\rho)$, for CAR and symmetric SAR models) is small relative to n. This typically occurs when $\boldsymbol{W}$ satisfies a large number of symmetries, in the sense of being invariant under a large group of permutations of its index set (see Biggs, 1993). An example is when $\boldsymbol{W}$ has constant off-diagonal entries and zero diagonal entries (see, e.g., Baltagi, 2006, and references therein), in which case $\boldsymbol{W}$ has only two eigenspaces, the one spanned by $\boldsymbol{\iota}$ and the hyperplane orthogonal to it. Thus, for a such $\boldsymbol{W}$, Condition B is satisfied whenever the entries of each regressor in the model sum to zero. Interestingly, if $\boldsymbol{X}$ contains an intercept, then $\mathrm{col}^{\perp}(\boldsymbol{X})$ is a subset of an eigenspace of $\boldsymbol{W}$, and thus the power function of a POI or LBI test is flat by Proposition 4.2 (cf. Theorem 5 of Kadiyala, 1970).

Remark G.7 Proposition 4.2 implies that, for CAR and symmetric SAR models satisfying Condition B, the envelope $\pi_{\rho}(\rho)$ is monotonic. One would expect the same property to hold also for zero-mean asymmetric SAR models, but, so far, we have found neither a proof nor a counterexample (by numerical analysis).

Proposition G.8 *Assume that Condition B holds. Then, in CAR and symmetric SAR models the limiting power of any POI and LBI test for $\rho = 0$ against $\rho > 0$ is 1 if $\boldsymbol{f}_{\max} \notin \mathrm{col}(\boldsymbol{X})$; strictly between α and 1 if $\boldsymbol{f}_{\max} \in \mathrm{col}(\boldsymbol{X})$ and $\mathrm{col}^{\perp}(\boldsymbol{X})$ is not a subset of an eigenspace of $\boldsymbol{W}$; α otherwise.*

Proof. Under Condition B, if $\boldsymbol{f}_i(\boldsymbol{W}) \notin \mathrm{col}(\boldsymbol{X})$, for $i = 1, ..., n$, then $\boldsymbol{f}_i(\boldsymbol{W}) \in \mathrm{col}^{\perp}(\boldsymbol{X})$. It follows that, if $\boldsymbol{f}_i(\boldsymbol{W}) \notin \mathrm{col}(\boldsymbol{X})$, for $i = 1, ..., n$, and $\boldsymbol{\Sigma}(\rho)$ is that of a CAR or symmetric SAR model, then $\boldsymbol{\Omega}_{\rho}\boldsymbol{C}\boldsymbol{f}_i(\boldsymbol{W}) = \boldsymbol{C}\boldsymbol{\Sigma}(\rho)\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{f}_i(\boldsymbol{W}) = \boldsymbol{C}\boldsymbol{\Sigma}(\rho)\boldsymbol{f}_i(\boldsymbol{W}) = \lambda_i(\boldsymbol{\Sigma}(\rho))\boldsymbol{C}\boldsymbol{f}_i(\boldsymbol{W})$, or, equivalently, $\{\boldsymbol{C}\boldsymbol{f}_i(\boldsymbol{W}),\ i \in H\}$, with H as defined in the proof of Proposition 4.1, is a set of $n-k$ orthogonal eigenvectors of $\boldsymbol{\Omega}_{\rho}$. Thus, in particular, $E_{n-k}(\boldsymbol{\Omega}_{\rho})$ does not depend on ρ. The proposition follows by Theorem 3.3, Lemma G.1 and Proposition 4.2. ∎

REFERENCES

Anderson, T.W. (1948) On the theory of testing serial correlation. Skandinavisk Aktuarietidskrift 31, 88-116.

Anselin, L. (1988) Spatial Econometrics: Methods and Models. Kluwer Academic Publishers.

Baltagi, B.H. (2006) Random effects and spatial autocorrelation with equal weights. Econometric Theory 22, 973-984.

Bartels, R., (1992) On the power function of the Durbin-Watson test. Journal of Econometrics 51, 101-112.

Berenblut, I.I. & G.I. Webb (1973) A new test for autocorrelated errors in the linear regression model. Journal of the Royal Statistical Society B, 33-50.

Besag, J.E. (1974) Spatial interaction and the statistical analysis of lattice data. Journal of the Royal Statistical Society B 36, 192-236.

Besag, J.E. & C. Kooperberg (1995) On conditional and intrinsic autoregression. Biometrika 82, 733-746.

Bhattacharyya, B.B., G.D. Richardson & L.A. Franklin (1997) Asymptotic inference for near unit roots in spatial autoregression. The Annals of Statistics 25, 1709-1724.

Biggs, N.L., (1993) Algebraic Graph Theory, second edition. Cambridge University Press.

Case, A., (1991) Spatial patterns in household demand. Econometrica 59, 953–966.

Cvetković, D.M., M. Doob & H. Sachs (1980) Spectra of Graphs. Academic Press.

Cliff, A.D. & J.K. Ord (1981) Spatial Processes: Models and Applications. Pion.

Cox, D.R. & D.V. Hinkley (1974) Theoretical Statistics. Chapman & Hall.

Durbin, J. (1970) An alternative to the bounds test for testing for serial correlation in least-squares regression. Econometrica 38, 422-29.

Cressie, N. (1993) Statistics for Spatial Data, revised edition. Wiley.

Fingleton, B. (1999) Spurious spatial regression: Some Monte Carlo results with a spatial unit root and spatial cointegration. Journal of Regional Science 39, 1-19.

Florax, R.J.G.M. & T. de Graaff (2004) The performance of diagnostic tests for spatial dependence in linear regression models: a meta-analysis of simulation studies. In L. Anselin, R.J.G.M. Florax & S. J. Rey (eds.), Advances in Spatial Econometrics: Methodology, Tools and Applications, pp. 29-65. Springer.

Gantmacher, F.R. (1974) The Theory of Matrices, vol. II. Chelsea.

Golub, G.H. & C.F. Van Loan (1996) Matrix Computations, third edition. The Johns Hopkins University Press.

Hardy, G., J.E. Littlewood & G. Pólya (1952) Inequalities, second edition. Cambridge University Press.

Horn, R. & C.R. Johnson (1985) Matrix Analysis. Cambridge University Press.

Hillier, G.H. (1987) Classes of similar regions and their power properties for some econometric testing problems. Econometric Theory 3, 1– 44.

Imhof, J.P. (1961) Computing the distribution of quadratic forms in normal variables. Biometrika 48, 419-26.

James, A.T. (1954) Normal multivariate analysis and the orthogonal group. Annals of Mathematical Statistics 25, 40-75.

Kadiyala, K.R. (1970) Testing for the independence of regression disturbances. Econometrica 38, 97-117.

Kariya, T. (1980) Locally robust tests for serial correlation in least squares regression. Annals of Statistics 8, 1065-1070.

Kelejian, H.H. & I.R. Prucha (2001) On the asymptotic distribution of the Moran I test statistic with applications. Journal of Econometrics 104, 219–257.

King, M.L. (1980) Robust tests for spherical symmetry and their application to least squares regression. Annals of Statistics 8, 1265-1271.

King, M.L. (1981) A small sample property of the Cliff-Ord test for spatial autocorrelation. Journal of the Royal Statistical Society B 43, 263-4.

King, M.L. (1988) Towards a theory of point optimal testing. Econometric Reviews 6, 169–255.

King, M.L. & G.H. Hillier (1985) Locally best invariant tests of the error covariance matrix of the linear regression model. Journal of the Royal Statistical Society B 47, 98-102.

Kleiber, C. & W. Krämer (2005) Finite-sample power of the Durbin-Watson test against fractionally integrated disturbances. Econometrics Journal 8, 406-417.

Kosfeld. R. & J. Lauridsen (2004) Dynamic spatial modelling of regional convergence processes. Empirical Economics 29, 705-22.

Krämer, W. (1985) The power of the Durbin-Watson test for regressions without an intercept. Journal of Econometrics 28, 363-370.

Krämer, W. (2005) Finite sample power of Cliff–Ord-type tests for spatial disturbance correlation in linear regression. Journal of Statistical Planning and Inference 128, 489-496.

Krämer, W. & and C. Donninger (1987) Spatial autocorrelation among errors and relative efficiency of OLS in the linear regression model. Journal of the American Statistical Association 82, 577-579.

Lauridsen, J. (2006) Spatial autoregressively distributed lag models: equivalent forms, estimation, and an illustrative commuting model. Annals of Regional Science 40, 297–311.

Lee, L.F. (2002) Consistency and efficiency of least squares estimation for mixed regressive, spatial autoregressive models. Econometric Theory 18, 252-277.

Lehmann, E.L. & J. Romano (2005) Testing Statistical Hypotheses, third edition. Springer.

Moran, P.A.P. (1950) Notes on continuos stochastic phenomena. Biometrika 37, 17-23.

Mur, J. & F.J. Trivez (2003) Unit roots and deterministic trends in spatial econometric models. International Regional Science Review 26, 289-312.

Ord, J.K. (1975) Estimation methods for models of spatial interaction. Journal of the American Statistical Association 70, 120-6.

Pace, R.K. & J.P. LeSage (2002) Semiparametric maximum likelihood estimates of spatial dependence. Geographical Analysis 34, 76-90.

Paulauskas, V. (2006) On unit roots for spatial autoregressive models. Journal of Multivariate Analysis, forthcoming.

Rahman, S. & M.L. King (1997) Marginal-likelihood score-based tests of regression disturbances in the presence of nuisance parameters. Journal of Econometrics 82, 81-106.

Tillman, J.A. (1975) The power of the Durbin-Watson test. Econometrica 43, 959-74.

Tunnicliffe Wilson, G. (1989) On the use of marginal likelihood in time series model estimation. Journal of the Royal Statistical Society B 51, 15-27.

Whittle, P. (1954) On stationary processes in the plane. Biometrika 41, 434-449.

Zeisel, H. (1989) On the power of the Durbin-Watson test under high autocorrelation. Communications in Statistics: Theory and Methods 18, 3907-3916.

	CAR	symmetric SAR	asymmetric SAR
$\boldsymbol{X} = \mathbf{O}$	1	1	either 0 or 1
$\boldsymbol{X} \neq \mathbf{O}$	in $[0, 1]$	in $[0, 1]$	in $[0, 1]$

Table 1: The limiting power of POI and LBI tests of $\rho = 0$ vs. $\rho > 0$ in CAR and SAR models, with and without regressors.

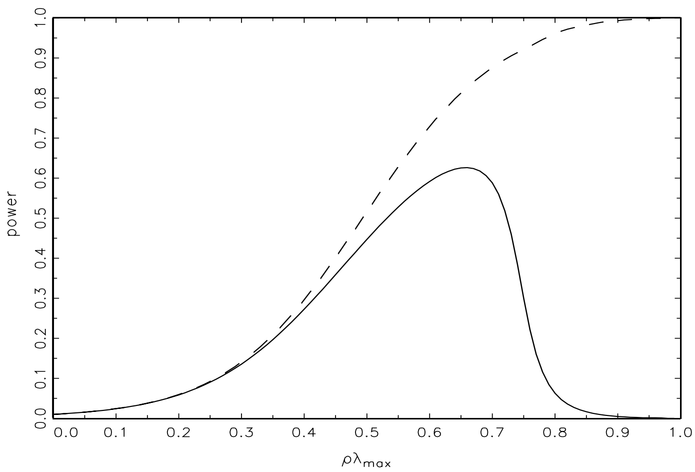

Figure 1: The power function of the Moran test (solid line) and the envelope $\pi_\rho(\rho)$ (dashed line) for the zero-mean asymmetric SAR model described in Example 3.11.

Table 2: Average correlation between neighbors (minimum and maximum correlation in parentheses) and percentage frequency distribution of the power $\pi_{LBI}(\rho)$ of the Cliff-Ord test, in model $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is a SAR process and $\boldsymbol{X}$ contains an intercept and a standard normal variate. The power is computed by the Imhof method over 10^6 replications of $\boldsymbol{X}$.

		av. neigh.	$\pi_{LBI}(\rho)$						
	$\rho\lambda_{\max}$	correlation	0.3-0.4	0.4-0.5	0.5-0.6	0.6-0.7	0.7-0.8	0.8-0.9	0.9-1
Nevada									
binary $\boldsymbol{W}$	0.90	0.85 (0.70−0.93)	0.11	0.25	28.42	71.05	0.17	·	·
	0.95	0.95 (0.87−0.98)	0.29	5.75	36.29	53.43	4.11	0.13	·
row-st $\boldsymbol{W}$	0.90	0.88 (0.81−0.93)	·	·	0.02	0.16	41.47	58.35	·
	0.95	0.96 (0.93−0.98)	·	·	0.01	0.05	1.56	98.38	·
Wyoming									
binary $\boldsymbol{W}$	0.90	0.80 (0.60−0.92)	·	·	·	0.02	0.69	99.29	·
	0.95	0.92 (0.77−0.98)	·	·	·	0.02	0.10	1.76	98.12
row-st $\boldsymbol{W}$	0.90	0.85 (0.76−0.92)	·	·	·	·	·	0.50	99.50
	0.95	0.95 (0.90−0.97)	·	·	·	·	·	·	100

Table 3: Probability of zero limiting power ($z_{0.05}$) and average shortcoming of the Cliff-Ord test, in the case of a binary $\boldsymbol{W}$.

		av. shortc. at $\rho\lambda_{\max} = 0.90$		av. shortc. at $\rho\lambda_{\max} = 0.95$	
	$z_{0.05}$	$\pi_{LBI}(\rho) \to 0$	$\pi_{LBI}(\rho) \to 1$	$\pi_{LBI}(\rho) \to 0$	$\pi_{LBI}(\rho) \to 1$
Nevada	0.77	0.20	0.16	0.32	0.24
Wyoming	$5.2 \cdot 10^{-4}$	0.15	0.03	0.26	0.02